ON A NONLINEAR PARTIAL DIFFERENTIAL ALGEBRAIC SYSTEM ARISING IN TECHNICAL TEXTILE INDUSTRY: ANALYSIS AND NUMERICS

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Abstract. In this paper a length-conserving numerical scheme for a nonlinear fourth order system of partial differential algebraic equations arising in technical textile industry is studied. Applying a semidiscretization in time, the resulting sequence of nonlinear elliptic systems with algebraic constraint is reformulated as constrained optimization problems in a Hilbert space setting that admit a solution at each time level. Stability and convergence of the scheme are proved. The numerical realization is performed by projected gradient methods on finite element spaces which determine the computational effort and approximation quality of the algorithm. Simulation results are presented and discussed in view of the application of an elastic inextensible fiber motion.

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1. Introduction

The numerical simulation and optimization of the dynamics of thin long elastic fibers are of great importance in the technical textile industry, for example in production processes of yarns or non-woven materials [16, 11]. In the slender-body theory a fiber can be asymptotically described by a parameterized, time-dependent curve that might represent the center-line. Then the fiber dynamics can be modeled by a nonlinear partial differential algebraic system (PDAEs) [13]. The partial differential equation of fourth order for the curve has a wavelike character with elliptic regularization. It can be considered as a reformulation of the Kirchhoff-Love equations for an elastic beam [12, 2]. The nonlinearity enters the system by the attachment of an algebraic constraint that incorporates inextensibility into the model and turns the inner traction to an unknown of the system, i.e. Lagrange multiplier. Depending on the application, also nonlinear or even stochastic source terms might play a role, see [15] and Figure 1.1. The studies of fiber lay-down processes and longtime behavior require a fast, accurate numerical treatment, cf. [11, 5, 8]. So far, the used approaches were mainly addressed to high-speed performance without any theoretical results on convergence or length conservation.

This work aims at the development of an appropriate numerical scheme for the PDAEs with focus on analytical and computational aspects. We propose a semi-discretization in time. Following the concept of [10] and employing a horizontal line method, we replace the transient problem by a sequence of elliptic systems that are handled in their weak formulation in terms of the Lagrange formalism. The algebraic constraint is incorporated in the definition of the optimization domain such that we study the solvability of a constrained minimization problem in a Hilbert space setting [9, 18]. We prove the existence of the minimizer and of the Lagrange multiplier on each time level. Stability estimates on the discrete solution and the Lagrange multiplier result then in the convergence of the numerical scheme. In addition, we derive an error bound on the fiber elongation. Numerically, we solve the optimization problems in finite element spaces by applying a projected gradient method with Armijo step size rule. Thereby, the finite dimensional approximation of the

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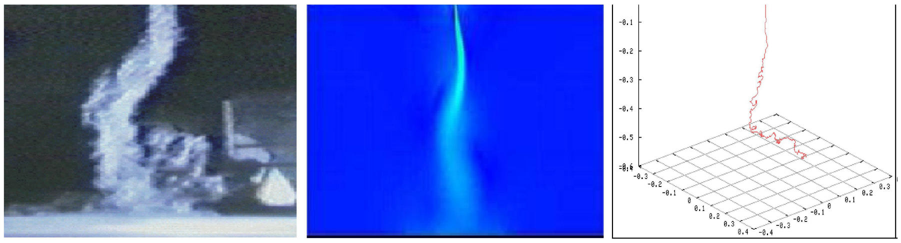

FIGURE 1.1. Application: melt-spinning process of non-woven materials. From left to right: Turbulent air flow in process (photo by industrial partner), mean velocity flow field, turbulence effects on immersed fiber modeled by stochastic forces (source terms) in fiber PDAEs (2.1), [14].

constraint and thus the choice of the projection mapping determine the efficiency and accuracy of the scheme.

The paper is structured as follows. We start with the presentation of the model equations in Section 2. After the introduction of the numerical scheme we deal with its theoretical analysis – regarding existence, stability and convergence – in Section 3. The numerical realization is discussed in view of computational effort and approximation quality in Section 4. The simulation results are in accordance with the analytical investigations and satisfy the predicted error estimates. We conclude with a summary and an outlook.

2. Model

Due to its slender geometry, a fiber can be asymptotically represented by its arclength-parameterized time-dependent center-line $\mathbf{r} : \Omega = \Omega_l \times \Omega_T \to \mathbb{R}^3$ according to the special Cosserat theory [2], where $\Omega_a := (0,a)$, $a \in (0,\infty)$ with fiber length l and end time T. Since extension and shear are here negligibly small in comparison to bending, the dynamics of an homogenous fiber can be described by a nonlinear partial differential equation of fourth order with an algebraic constraint for the conservation of length [13]

$$\omega\, \partial_{tt}\mathbf{r}(s,t) = \partial_s(\lambda(s,t)\partial_s\mathbf{r}(s,t)) - b\, \partial_{ssss}\mathbf{r}(s,t) + \mathbf{f}(\mathbf{r}(\cdot),s,t) \tag{2.1a}$$

$$|\partial_s\mathbf{r}(s,t)|^2 = 1 \tag{2.1b}$$

where $\omega > 0$ denotes the line weight. The dynamics is caused by acting outer and inner forces (Newton's law). The outer force densities $\mathbf{f}$ might come for example from gravity and/or aerodynamics regarding the application. The inner force densities stem from traction λ and bending with bending stiffness $b > 0$. The inner traction $\lambda : \Omega \to \mathbb{R}$ might be interpreted as Lagrange multiplier to the algebraic constraint (2.1b) that is expressed in the Euclidian norm $|\cdot|$. In addition, torsion could be included, yielding an extra term $\kappa(\partial_s\mathbf{r} \times \partial_{sss}\mathbf{r})$ in (2.1a) with $\partial_s\kappa = 0$, [11]. Its neglect ($\kappa = 0$) is here justified by the set-up that is characterized by a free fiber ending. In particular, we consider a fiber fixed at one ending ($s = l$) that is freely swinging. This set-up is clearly a simplification to real applications in technical textile industry, but it still contains the major mathematical difficulty, i.e. the partial differential-algebraic structure of the model equations. Then, Dirichlet and Neumann boundary conditions for the fixed and stress-free fiber ending respectively as well as appropriate initial conditions close (2.1) to an initial boundary value problem

$$\begin{aligned}
\mathbf{r}(l,t) &= \mathbf{0}, & \partial_s\mathbf{r}(l,t) &= -\mathbf{e_g}, & & \\
\partial_{ss}\mathbf{r}(0,t) &= \mathbf{0}, & \partial_{sss}\mathbf{r}(0,t) &= \mathbf{0}, & \lambda(0,t) &= 0, \\
\mathbf{r}(s,0) &= (l-s)\mathbf{e_g}, & \partial_t\mathbf{r}(s,0) &= \mathbf{0}. & &
\end{aligned} \tag{2.2}$$

with unit vector $\mathbf{e_g}$ prescribing the direction of gravity.

The fiber model (2.1a) is a wavelike system with elliptic regularization. It is a reformulation of the Kirchhoff-Love equations for an elastic beam [12]. The mathematical challenge lies in the treatment of the algebraic constraint of inextensibility (2.1b). This paper aims at the derivation and investigation of a numerical scheme for the partial differential algebraic system. For simplicity, we restrict here to sufficiently smooth outer forces that are independent of the fiber position, $\mathbf{f}:\Omega\to\mathbb{R}^3$, like gravity. However, note that in melt-spinning processes of non-woven materials, even stochastic forces from turbulent air flows become important [15, 14], cf. Figure 1.1. For a study on stochastic extensible beam equations see e.g. [6, 3].

3. Theoretical analysis

In this section we derive an implicit semi-discretization for the partial differential algebraic fiber system (2.1). Following the concept of [10] and employing a horizontal line method (Rothe method) in time, we replace the transient problem by a sequence of elliptic systems that are handled in their weak formulation in terms of the Lagrange formalism. Hereby, we incorporate the algebraic constraint in the definition of the optimization domain such that we study the solvability of a constrained minimization problem. In particular, we show the existence of the minimizer and of the Lagrange multiplier on each time level. Stability estimates on the discrete solution and the Lagrange multiplier result then in a convergence proof for the numerical scheme.

3.1. **Semi-discretization.** Let $T\in(0,\infty)$ be given. We divide the time interval $[0,T]$ into N subintervals by introducing the temporal mesh $\{t_k\,|\,k=0,...,N\}$ where $t_k=k\tau$ is prescribed by the time step $\tau=T/N$. Instead of the taken equidistant discretization one can also think of a varying time step $\tau_k=t_k-t_{k-1}$ and define the maximal subinterval length $\tau=\max_{k=1,\dots,N}\tau_k$. Then the partition is assumed to satisfy $\tau\to 0$ as $N\to\infty$.

Using an implicit Euler scheme, we discretize the fiber system as

$$\omega\left(\frac{\mathbf{r}_{k+1}-2\mathbf{r}_k+\mathbf{r}_{k-1}}{\tau^2}\right)=\partial_s(\lambda_{k+1}\partial_s\mathbf{r}_{k+1})-b\partial_{ssss}\mathbf{r}_{k+1}+\mathbf{f}_{k+1} \tag{3.1a}$$

$$|\partial_s\mathbf{r}_{k+1}|^2=1, \tag{3.1b}$$

with $\mathbf{f}_{k+1}=\mathbf{f}(t_{k+1})$, $k=1,\dots,N-1$, $\mathbf{r}_0=(l-s)\mathbf{e_g}$ and $\mathbf{r}_1=\mathbf{r}_0$. The implicit time discretization requires consequently the recursive solving of nonlinear elliptic systems in one space dimension. The approximate solution to (2.1) is then given by the linear interpolation $(\mathbf{r}^\tau,\lambda^\tau)$, i.e.,

$$\mathbf{r}^\tau(s,t)=\frac{t-t_{k-1}}{\tau}(\mathbf{r}_k-\mathbf{r}_{k-1})+\mathbf{r}_{k-1},\qquad s\in\Omega_l,\quad t\in(t_{k-1},t_k],\quad k=1,\dots,N,$$

and correspondingly for λ^τ, where $\lambda_0=\lambda_1=0$. For functions defined on $[0,T]$, in turn, a subindex $k\in\{0,\dots,N\}$ corresponds to the value of the function at time t_k.

The discretized system (3.1) can be identified as Euler-Lagrange equations corresponding to the appropriate Lagrange functional such that we study its solvability as variational problem in the following.

3.2. **Solvability of discretized system.** The norm of a Banach space B we denote by $\|\cdot\|_B$ and the dual pairing with its dual space B' by ${}_B\langle\cdot,\cdot\rangle_{B'}$. If B even is a Hilbert space, then its inner product we denote by $(\cdot,\cdot)_B$. By $L^2(S;\mathbb{R}^n)$, $S\subset\mathbb{R}^d$ Lebesgue measurable, $n,d\in\mathbb{N}$, we denote the Hilbert space of (equivalence classes of) square integrable functions on S w.r.t. the Lebesgue measure taking values in $\mathbb{R}^n$. The space $C^0(S;B)$ of continuous functions on compact S with values in B we consider to be equipped with the norm of uniform convergence. We use the notation $W^{m,p}(U;\mathbb{R}^n)$, $U\subset\mathbb{R}^d$ open, $m\in[0,\infty)$, $p\in[1,\infty]$, for Sobolev spaces as in [1]. In case of $p=2$, the Hilbert space $W^{m,2}(U;\mathbb{R}^n)$ is abbreviated by $H^m(U;\mathbb{R}^n)$. In the case $n=1$ we suppress the range of function spaces. In particular, we introduce the notation

$$H^m_{0,a}(\Omega_a;\mathbb{R}^n):=\{\mathbf{v}\in H^m(\Omega_a;\mathbb{R}^n)\,|\,\partial_s^\alpha\mathbf{v}(a)=\mathbf{0}\text{ for all }\alpha\in\mathbb{N}_0,\alpha+1/2<m\},\quad \Omega_a=(0,a),$$

$a\in(0,\infty)$. Of course, $H^m_{0,a}(\Omega_a;\mathbb{R}^n)$ equipped with the norm of $H^m(\Omega_a;\mathbb{R}^n)$ is a Hilbert space. Its dual space $(H^m_{0,a}(\Omega_a;\mathbb{R}^n))'$ we denote by $H^{-m}(\Omega_a;\mathbb{R}^n)$. Recall that $H^m(\Omega_a;\mathbb{R}^n)$ is embedded continuously and compactly in the Hölder spaces $C^{k,\gamma}([0,a];\mathbb{R}^n)$ for $m>1/2+k+\gamma$, $k\in\mathbb{N}_0$,

$0 \leq \gamma \leq 1$, see e.g. [1]. We always, via the Riesz representation theorem, identify spaces of square integrable functions with their dual space and consider an embedding of Sobolev spaces in the sense of Gelfand triples with the space of square integrable functions as central space.

Then, we define the affine linear fiber space

$$V := \{\mathbf{v} \in \mathbf{v}_D + H^2_{0,l}(\Omega_l;\mathbb{R}^3) \mid \mathbf{v}_D \in H^2(\Omega_l;\mathbb{R}^3),\ \mathbf{v}_D(l) = \mathbf{0},\ \partial_s \mathbf{v}_D(l) = -\mathbf{e_g}\}. \tag{3.2}$$

We consider the constraint of inextensibility (3.1b) as $\partial_t|\partial_s \mathbf{r}_{k+1}|^2 = 0$ and express it in terms of

$$e_{k+1} : V \to H^1_{0,l}(\Omega_l), \quad e_{k+1}(\mathbf{v}) = 2\partial_s(\mathbf{v} - \mathbf{r}_k) \cdot \partial_s \mathbf{r}_k = 0. \tag{3.3}$$

Moreover, we deduce the cost functionals $J_{k+1} : V \to \mathbb{R}$

$$J_{k+1}(\mathbf{v}) = \omega \left\| \tau \mathrm{D}^2_{k+1}\mathbf{v}\right\|^2_{L^2(\Omega_l)} + b\,\|\partial_{ss}\mathbf{v}\|^2_{L^2(\Omega_l)} - 2\,(\mathbf{f}_{k+1}, \mathbf{v})_{L^2(\Omega_l)} \tag{3.4}$$

with the second temporal difference $\mathrm{D}^2_{k+1}\mathbf{v} = (\mathbf{v} - 2\mathbf{r}_k + \mathbf{r}_{k-1})/\tau^2$ by applying variational calculus on (3.1a) for $k = 1, ..., N-1$.

Lagrange formalism. *For $k = 1, ..., N-1$, let J_{k+1} be the cost functional of* (3.4) *and e_{k+1} the constraint functional of* (3.3). *Define the Lagrange functional $L_{k+1} : V \times H^{-1}(\Omega_l) \to \mathbb{R}$ by*

$$L_{k+1}(\mathbf{v}, \lambda) = J_{k+1}(\mathbf{v}) + {}_{H^1_{0,l}(\Omega_l)}\langle e_{k+1}(\mathbf{v}), \lambda\rangle_{H^{-1}(\Omega_l)}.$$

Then, the minimizer of the Lagrange functional is a weak solution of the fiber system (3.1).

The minimizer of the Lagrange functional satisfies the adjoint problem (3.5) for all test functions $\eta \in H^{-1}(\Omega_l)$ and $\boldsymbol{\phi} \in H^2_{0,l}(\Omega_l, \mathbb{R}^3)$, i.e.

$$\partial_\lambda L_{k+1}(\mathbf{v}, \lambda)[\eta] = 0 = {}_{H^1_{0,l}(\Omega_l)}\langle e_{k+1}(\mathbf{v}), \eta\rangle_{H^{-1}(\Omega_l)} \tag{3.5a}$$

$$\begin{aligned}\nabla_{\mathbf{v}} L_{k+1}(\mathbf{v}, \lambda)[\boldsymbol{\phi}] = 0 &= J'_{k+1}(\mathbf{v})[\boldsymbol{\phi}] + {}_{H^1_{0,l}(\Omega_l)}\langle e'_{k+1}(\mathbf{v})[\boldsymbol{\phi}], \lambda\rangle_{H^{-1}(\Omega_l)} \\ &= 2\big(\omega\,(\mathrm{D}^2_{k+1}\mathbf{v}, \boldsymbol{\phi})_{L^2(\Omega_l)} + b\,(\partial_{ss}\mathbf{v}, \partial_{ss}\boldsymbol{\phi})_{L^2(\Omega_l)} - (\mathbf{f}_{k+1}, \boldsymbol{\phi})_{L^2(\Omega_l)} \\ &\quad + {}_{H^1_{0,l}(\Omega_l)}\langle \partial_s \mathbf{r}_k \cdot \partial_s \boldsymbol{\phi}, \lambda\rangle_{H^{-1}(\Omega_l)}\big).\end{aligned} \tag{3.5b}$$

Presupposing sufficient regularity of the Lagrange multiplier λ, the duality pairing ${}_{H^1_{0,l}(\Omega_l)}\langle\cdot,\cdot\rangle_{H^{-1}(\Omega_l)}$ coincides with $(\cdot,\cdot)_{L^2(\Omega_l)}$ in the sense of a Gelfand triple. This yields the Euler-Lagrange equations corresponding to the fiber system. Hence, the weak solvability of the elliptic fiber system (3.1) can be formulated as

Constrained minimization problem

Minimize J_{k+1} over the domain $K_{k+1} := \{\mathbf{v} \in V \mid e_{k+1}(\mathbf{v}) = 0$ and $|\partial_s \mathbf{r}_k - \partial_s \mathbf{v}| \leq \tau^2\}$. (3.6)

Lemma 1 (Properties of cost functional)**.** *For $k = 1, ..., N-1$, the cost functional $J_{k+1} : V \to \mathbb{R}$ defined in* (3.4) *is strictly convex, coercive and weakly lower semi-continuous. The minimization domain K_{k+1} is closed and convex and, in particular, weakly closed.*

Proof. Here and throughout the following proofs where is no danger of confusion, we suppress the indices indicating the time levels for a simpler notation. Let $\mathbf{u}, \mathbf{v} \in V$, $\mathbf{u} \neq \mathbf{v}$, $\mu \in (0,1)$. Then, the strict convexity of J is concluded from

$$\begin{aligned}&\mu J(\mathbf{u}) + (1-\mu)J(\mathbf{v}) - J(\mu\mathbf{u} + (1-\mu)\mathbf{v}) \\ &\quad = (\mu - \mu^2)(\omega\,\|\tau\mathrm{D}^2\mathbf{u} - \tau\mathrm{D}^2\mathbf{v}\|^2_{L^2(\Omega_l)} + b\,\|\partial_{ss}\mathbf{u} - \partial_{ss}\mathbf{v}\|^2_{L^2(\Omega_l)}) > 0\end{aligned}$$

since $\omega, b > 0$.

Due to the assumed boundary conditions a Poincaré inequality holds and we obtain

$$\omega\|\tau\mathrm{D}^2\mathbf{v}\|^2_{L^2(\Omega_l)} + b\|\partial_{ss}\mathbf{v}\|^2_{L^2(\Omega_l)} \geq A\|\mathbf{v}\|^2_{H^2(\Omega_l)} - B$$

for some $0 < A, B < \infty$. Hence

$$\begin{aligned}J(\mathbf{v}) &= \omega\|\tau \mathrm{D}^2\mathbf{v}\|^2_{L^2(\Omega_l)} + b\|\partial_{ss}\mathbf{v}\|^2_{L^2(\Omega_l)} - 2\langle \mathbf{f}, \mathbf{v}\rangle_{L^2(\Omega_l)} \geq A\|\mathbf{v}\|^2_{H^2(\Omega_l)} - 2|\langle \mathbf{f}, \mathbf{v}\rangle_{L^2(\Omega_l)}| - B \\ &\geq \|\mathbf{v}\|_{H^2(\Omega_l)}\left(A\|\mathbf{v}\|_{H^2(\Omega_l)} - 2\|\mathbf{f}\|_{L^2(\Omega_l)}\right) - B.\end{aligned}$$

Thus, $J(\mathbf{v}) \to \infty$, if $\|\mathbf{v}\|_{H^2(\Omega_l)} \to \infty$ for fixed $\mathbf{f} \in L^2(\Omega_l)$, i.e., J is coercive.

Let $(\mathbf{v}_n)_{n\in\mathbb{N}}$ be a sequence in V that converges weakly to $\mathbf{v} \in V$ in H^2, i.e., $\mathbf{v}_n \overset{H^2}{\rightharpoonup} \mathbf{v}$ for $n \to \infty$. Then, in particular, $\mathbf{v}_n \overset{L^2}{\rightharpoonup} \mathbf{v}$ and $\partial_{ss}\mathbf{v}_n \overset{L^2}{\rightharpoonup} \partial_{ss}\mathbf{v}$ for $n \to \infty$. Since the norm is lower semi-continuous w.r.t. weak convergence and the inner product with $\mathbf{f} \in L^2(\Omega_l)$ is continuous w.r.t. weak convergence, we obtain $J(\mathbf{v}) \leq \lim_{n\to\infty} \inf J(\mathbf{v}_n)$, i.e., J is weakly lower semi-continuous.

The convexity of K results from the affine linearity of e and the convexity of the set $\{\mathbf{v} \in H^2(\Omega_l;\mathbb{R}^3) \,|\, |\partial_s \mathbf{r}_k - \partial_s \mathbf{v}| \leq \tau^2\}$. Let $\mathbf{u}$, $\mathbf{v} \in K$, $\mu \in [0,1]$, then $\mu\mathbf{u} + (1-\mu)\mathbf{v} \in K$, since

$$e(\mu\mathbf{u} + (1-\mu)\mathbf{v}) = \mu e(\mathbf{u}) + (1-\mu)e(\mathbf{v}) = 0 \quad \text{and}$$
$$|\partial_s\mathbf{r}_k - \partial_s(\mu\mathbf{u} + (1-\mu)\mathbf{v})| \leq \mu|\partial_s\mathbf{r}_k - \partial_s\mathbf{u}| + (1-\mu)|\partial_s\mathbf{r}_k - \partial_s\mathbf{v}| \leq \tau^2.$$

Since V is closed, e is continuous and the set $\{\mathbf{v} \in H^2(\Omega_l;\mathbb{R}^3) \,|\, |\partial_s\mathbf{r}_k - \partial_s\mathbf{v}| \leq \tau^2\}$ is closed, also K is closed. This, together with convexity, implies that K is also weakly closed. □

Theorem 2 (Existence and uniqueness of minimizer)**.** *The constrained minimization problem* (3.6) *has a unique solution* $\mathbf{r}_{k+1} \in K_{k+1}$ *on every time level.*

Proof. According to Lemma 1 the cost functional J on every time level is coercive and weakly lower semi-continuous and the domain K is convex and weakly closed. These are the requirements for a general existence and uniqueness result for constrained minimization problems, see e.g. [18]. We state the proof here for completeness.

Choose a minimizing sequence $(\mathbf{v}_n)_{n\in\mathbb{N}}$, $\mathbf{v}_n \in K$, with $J(\mathbf{v}_n) \to \inf_{\mathbf{v}\in K} J(\mathbf{v})$ for $n \to \infty$. Then $-\infty < \inf_{\mathbf{v}\in K} J(\mathbf{v}) < \infty$ and $(\mathbf{v}_n)_{n\in\mathbb{N}}$ is bounded in view of the coercivity of J. Hence, there exists a subset $N \subset \mathbb{N}$ and $\mathbf{r} \in V$ such that $\mathbf{v}_n \overset{H^2}{\rightharpoonup} \mathbf{r}$ for $N \ni n \to \infty$. Since K is weakly closed, $\mathbf{r} \in K$. The weak lower semi-continuity of J implies

$$J(\mathbf{r}) \leq \inf_{n\in N} J(\mathbf{v}_n),$$

whence $\mathbf{r}$ is a minimizer.

Since K is convex, the strict convexity of J on V implies the uniqueness of the minimizer. Assume $\mathbf{u}, \mathbf{v} \in K$ to be two minimizers, $\mathbf{u} \neq \mathbf{v}$ with $J(\mathbf{u}) = J(\mathbf{v})$. Then $J(\mu\mathbf{u} + (1+\mu)\mathbf{v}) < \mu J(\mathbf{u}) + (1-\mu)J(\mathbf{v}) = J(\mathbf{u})$ for $\mu \in (0,1)$. Since $\mu\mathbf{u} + (1+\mu)\mathbf{v} \in K$ for $\mu \in (0,1)$, this contradicts the assumption. □

Note that the uniqueness of the minimizer is meaningless for the solvability statement of the fiber system, since the unique minimizer need not necessarily be the only solution in view of possibly existing saddle points.

Since

$$|\partial_s\mathbf{r}_{k+1}| \leq |\partial_s\mathbf{r}_{k+1} - \partial_s\mathbf{r}_k| + |\partial_s\mathbf{r}_k| \leq \tau^2 + |\partial_s\mathbf{r}_k - \partial_s\mathbf{r}_{k-1}| + |\partial_s\mathbf{r}_{k-1}| \leq \ldots \leq N\tau^2 + |\mathbf{e_g}| = 1 + T\tau$$

and $e_{k+1}(\mathbf{r}_{k+1}) = 0$ for all $k = 1, ..., N-1$, the following lemma follows immediately by the Cauchy–Schwarz inequality.

Lemma 3. *The relation* $1 \leq |\partial_s\mathbf{r}_k(s)| \leq |\partial_s\mathbf{r}_{k+1}(s)| \leq 1 + T\tau$ *holds for all* $s \in \Omega_l$, $k = 0, ..., N-1$.

Proposition 4 (Surjectivity of linearized constraint functional)**.** *For* $k = 1, ..., N-1$ *the linearized constraint functional* $e'_{k+1} \in L(H^2_{0,l}(\Omega_l;\mathbb{R}^3); H^1_{0,l}(\Omega_l))$ *is surjective.*

Proof. We have

$$e'_{k+1}[\boldsymbol{\phi}] = 2\partial_s\mathbf{r}_k \cdot \partial_s\boldsymbol{\phi}, \quad \boldsymbol{\phi} \in H^2_{0,l}(\Omega_l).$$

Let $\psi \in H^1_{0,l}(\Omega_l)$ be arbitrary. Set

$$\boldsymbol{\phi}(s) = -\int_s^l \frac{\psi\,\partial_u \mathbf{r}_k}{2|\partial_u \mathbf{r}_k|^2}\,du, \quad s \in \Omega_l. \tag{3.7}$$

Note that $\partial_s \mathbf{r}_k, \psi \in C^0([0,l])$. Hence, together with Lemma 3, we obtain $\boldsymbol{\phi} \in L^2(\Omega_l;\mathbb{R}^3)$ with $\boldsymbol{\phi}(l) = 0$. Furthermore

$$\partial_s \boldsymbol{\phi} = \frac{\psi\,\partial_s \mathbf{r}_k}{2|\partial_s \mathbf{r}_k|^2}$$

and therefore

$$\psi = 2\partial_s \mathbf{r}_k \cdot \partial_s \boldsymbol{\phi} = e'_{k+1}[\boldsymbol{\phi}].$$

Moreover $\partial_s \boldsymbol{\phi} \in L^2(\Omega_l;\mathbb{R}^3)$ with $\partial_s \boldsymbol{\phi}(l) = 0$. Finally, $\partial_{ss} \boldsymbol{\phi} \in L^2(\Omega_l;\mathbb{R}^3)$ can be concluded from $\partial_{ss} \mathbf{r}_k \in L^2(\Omega_l;\mathbb{R}^3)$, $\partial_s \psi, \psi \in L^2(\Omega_l)$, $\partial_s \mathbf{r}_k \in L^\infty(\Omega_l;\mathbb{R}^3)$ and $\psi \in L^\infty(\Omega_l)$ using the chain rule. This shows $\boldsymbol{\phi} \in H^2_{0,l}(\Omega_l)$. □

Remark 5. *Note that e'_{k+1} is not injective on $H^2_{0,l}(\Omega_l;\mathbb{R}^3)$. Nevertheless we denote the mapping $\psi \mapsto \boldsymbol{\phi}$ in (3.7) by $(e'_{k+1})^{-1}$, because $e'_{k+1}(e'_{k+1})^{-1}$ is the identity on $H^1_{0,l}(\Omega_l)$. Of course, $(e'_{k+1})^{-1} \in L(H^1_{0,l}(\Omega_l); H^2_{0,l}(\Omega_l;\mathbb{R}^3))$ by the inverse mapping theorem (applied in the proper quotient space setting).*

Theorem 6 (Existence of discrete solution)**.** *For $k = 1, ..., N-1$ let $\mathbf{r}_{k+1}$ be the minimizer of J_{k+1} on K_{k+1}, provided in Theorem 2, and*

$$\lambda_{k+1} := -J'_{k+1}(\mathbf{r}_{k+1})(e'_{k+1})^{-1}. \tag{3.8}$$

Then $\lambda_{k+1} \in H^{-1}(\Omega_l)$ (see Remark 5), and $(\mathbf{r}_{k+1}, \lambda_{k+1})$ are solving weakly the discrete fiber system (3.1)*, i.e.,*

$$\begin{aligned}
\omega\,(\mathrm{D}^2_{k+1}\mathbf{r}_{k+1}, \boldsymbol{\phi})_{L^2(\Omega_l)} &= -{}_{H^1_{0,l}(\Omega_l)}\langle \partial_s \mathbf{r}_k \cdot \partial_s \boldsymbol{\phi}, \lambda_{k+1}\rangle_{H^{-1}(\Omega_l)} \\
&\qquad - b\,(\partial_{ss}\mathbf{r}_{k+1}, \partial_{ss}\boldsymbol{\phi})_{L^2(\Omega_l)} + (\mathbf{f}_{k+1}, \boldsymbol{\phi})_{L^2(\Omega_l)} &(3.9a)\\
\partial_s(\mathbf{r}_{k+1} - \mathbf{r}_k) \cdot \partial_s \mathbf{r}_k &= 0. &(3.9b)
\end{aligned}$$

for all test functions $\boldsymbol{\phi} \in H^2_{0,l}(\Omega_l;\mathbb{R}^3)$.

Proof. By definition, all elements from K_{k+1} fulfill (3.9b). Furthermore, since $\mathbf{r}_{k+1}$ minimizes J_{k+1} on K_{k+1} and

$$e_{k+1}[\boldsymbol{\phi} - (e'_{k+1})^{-1}[2\partial_s \mathbf{r}_k \cdot \partial_s \boldsymbol{\phi}]] = 0 \quad \text{for all } \boldsymbol{\phi} \in H^2_{0,l}(\Omega_l;\mathbb{R}^3),$$

we have

$$\begin{aligned}
0 &= J'_{k+1}(\mathbf{r}_{k+1})[\boldsymbol{\phi} - (e'_{k+1})^{-1}[2\partial_s \mathbf{r}_k \cdot \partial_s \boldsymbol{\phi}]] \\
&= 2\big(\omega\,(\mathrm{D}^2_{k+1}\mathbf{r}_{k+1}, \boldsymbol{\phi})_{L^2(\Omega_l)} + b\,(\partial_{ss}\mathbf{r}_{k+1}, \partial_{ss}\boldsymbol{\phi})_{L^2(\Omega_l)} - (\mathbf{f}_{k+1}, \boldsymbol{\phi})_{L^2(\Omega_l)} \\
&\quad + {}_{H^1_{0,l}(\Omega_l)}\langle \partial_s \mathbf{r}_k \cdot \partial_s \boldsymbol{\phi}, \lambda_{k+1}\rangle_{H^{-1}(\Omega_l)}\big) \qquad \text{for all } \boldsymbol{\phi} \in H^2_{0,l}(\Omega_l;\mathbb{R}^3).
\end{aligned}$$

□

3.3. **Stability estimates.** In the following we provide stability estimates for the discrete solution. For function spaces $B_1(S_1)$ and $B_2(S_2)$ on sets S_1 and S_1, respectively, we define as usual $B_1(S_1) \otimes B_2(S_2) := span\{f_1 f_2 \,|\, f_1 \in B_1(S_1), f_2 \in B_2(S_2)\}$, where $(f_1 f_2)(s_1, s_2) := f_1(s_1) f_2(s_2)$, $s_1 \in S_1$, $s_2 \in S_2$ (algebraic tensor product). The Sobolev space $H^{2,1}(\Omega;\mathbb{R}^3)$ on $\Omega := \Omega_l \times \Omega_T$ then is defined as the completion of $H^2(\Omega_l;\mathbb{R}^3) \otimes H^1(\Omega_T)$ w.r.t. the metric associated to its inner product, see e.g. [17, Chap. II.4] (i.e., it is the Sobolev space of functions on Ω which are twice weakly differentiable in the first variable and once weakly differentiable in the second variable and square integrable on Ω together with their derivatives). Correspondingly, we set $H^{m_l,m_T}_{0,l,T}(\Omega)$, $m_l, m_T \in [0,\infty)$, to be the completion of $H^{m_l}_{0,l}(\Omega_l) \otimes H^{m_T}_{0,T}(\Omega_T)$ and use the notation $H^{-m_l,-m_T}(\Omega) := (H^{m_l,m_T}_{0,l,T}(\Omega))'$. In the

case $m_l = m_T$ we suppress the index m_T. For functions h defined on Ω_T we use the following notation for discrete derivatives:

$$(\mathrm{D}h)_k = (\mathrm{D}^1 h)_k := \frac{h_k - h_{k-1}}{\tau}, \quad (\mathrm{D}^n h)_k := (\mathrm{D}(\mathrm{D}^{n-1}h)^\tau)_k, \quad k = n, \ldots, N.$$

Proposition 7 (Stability estimates for $\mathbf{r}^\tau$). *Let $\mathbf{r}_{k+1} \in V$ be as in Theorem 6, $k = 1, ..., N-1$, and let $\mathbf{r}^\tau \in H^{2,1}(\Omega;\mathbb{R}^3)$ be the corresponding linear interpolation. Then there exists $0 < K < \infty$, independent of $N \in \mathbb{N}$ (or, equivalently, the time discretization $\tau > 0$), such that*

$$\max_{1\le k\le N} \|(\mathrm{D}\mathbf{r}^\tau)_k\|_{L^2(\Omega_l)} \le K, \quad \max_{0\le k\le N} \|\partial_{ss}\mathbf{r}_k\|_{L^2(\Omega_l)} \le K, \quad \textit{and} \quad \|\mathbf{r}^\tau\|_{H^{2,1}(\Omega)} \le K. \tag{3.10}$$

Proof. Since $\mathbf{r}_{k+1} \in V \subset H^2(\Omega_l;\mathbb{R}^3)$ for all $k = 1, ..., N-1$ and $\mathbf{r}^\tau(s)$ is piecewise linear for all $s \in \Omega_l$, we have $\mathbf{r}^\tau \in H^{2,1}(\Omega;\mathbb{R}^3)$. We know that

$$\begin{aligned}\omega\left((\mathrm{D}^2\mathbf{r}^\tau)_{k+1}, \boldsymbol{\phi}\right)_{L^2(\Omega_l)} = &-{}_{H^1(\Omega_l)}\langle \partial_s \mathbf{r}_k \cdot \partial_s \boldsymbol{\phi}, \lambda_{k+1}\rangle_{H^{-1}(\Omega_l)} \\ &- b\left(\partial_{ss}\mathbf{r}_{k+1}, \partial_{ss}\boldsymbol{\phi}\right)_{L^2(\Omega_l)} + (\mathbf{f}_{k+1}, \boldsymbol{\phi})_{L^2(\Omega_l)},\end{aligned}$$

for all $\boldsymbol{\phi} \in H^2_{0,l}(\Omega_l;\mathbb{R}^3)$. Note that the first summand on the right-hand side is discretized in an explicit way. Since $(\mathrm{D}^2\mathbf{r}^\tau)_{k+1} = ((\mathrm{D}\mathbf{r}^\tau)_{k+1} - (\mathrm{D}\mathbf{r}^\tau)_k)/\tau$, the special choice $\boldsymbol{\phi} = \mathbf{r}_{k+1} - \mathbf{r}_k \in H^2_{0,l}(\Omega_l;\mathbb{R}^3)$ results in

$$\begin{aligned}&\omega\left(\|(\mathrm{D}\mathbf{r}^\tau)_{k+1}\|^2_{L^2(\Omega_l)} - \|(\mathrm{D}\mathbf{r}^\tau)_k\|^2_{L^2(\Omega_l)} + \|\tau(\mathrm{D}^2\mathbf{r}^\tau)_{k+1}\|^2_{L^2(\Omega_l)}\right) \\ &\quad = -b\left(\|\partial_{ss}\mathbf{r}_{k+1}\|^2_{L^2(\Omega_l)} - \|\partial_{ss}\mathbf{r}_k\|^2_{L^2(\Omega_l)} + \|\partial_{ss}(\mathbf{r}_{k+1} - \mathbf{r}_k)\|^2_{L^2(\Omega_l)}\right) + 2(\mathbf{f}_{k+1}, \mathbf{r}_{k+1} - \mathbf{r}_k)_{L^2(\Omega_l)}\end{aligned}$$

by applying the identity $2(a-b)a = a^2 - b^2 + (a-b)^2$ and the functional constraint $e_{k+1}(\mathbf{r}_{k+1}) = 0$. Hence, we obtain

$$\begin{aligned}&\omega\|(\mathrm{D}\mathbf{r}^\tau)_{k+1}\|^2_{L^2(\Omega_l)} + b\|\partial_{ss}\mathbf{r}_{k+1}\|^2_{L^2(\Omega_l)} \\ &\qquad\qquad \le \omega\|(\mathrm{D}\mathbf{r}^\tau)_k\|^2_{L^2(\Omega_l)} + b\|\partial_{ss}\mathbf{r}_k\|^2_{L^2(\Omega_l)} + 2\tau(\mathbf{f}_{k+1}, (\mathrm{D}\mathbf{r}^\tau)_{k+1})_{L^2(\Omega_l)}.\end{aligned}$$

Summing up $k = 1, \ldots, M-1 \le N-1$ gives the following crucial relation

$$\omega\|(\mathrm{D}\mathbf{r}^\tau)_M\|^2_{L^2(\Omega_l)} + b\|\partial_{ss}\mathbf{r}_M\|^2_{L^2(\Omega_l)} \le 2\tau \sum_{k=1}^{M-1} (\mathbf{f}_{k+1}, (\mathrm{D}\mathbf{r}^\tau)_{k+1})_{L^2(\Omega_l)}, \tag{3.11}$$

(note that $(\mathrm{D}\mathbf{r}^\tau)_1 = \mathbf{0}$, as well as, $\partial_{ss}\mathbf{r}_1 = \mathbf{0}$). We estimate the scalar product on the right-hand side by Cauchy–Schwarz and Young's inequality, i.e. $2ab \le a^2 + b^2$, and find

$$\|(\mathrm{D}\mathbf{r}^\tau)_M\|^2_{L^2(\Omega_l)} \le \frac{\tau}{\omega}\left(\sum_{k=1}^{N-1} \|\mathbf{f}_{k+1}\|^2_{L^2(\Omega_l)} + \sum_{k=1}^{M-1} \|(\mathrm{D}\mathbf{r}^\tau)_{k+1}\|^2_{L^2(\Omega_l)}\right).$$

The discrete Gronwall Lemma implies

$$\|(\mathrm{D}\mathbf{r}^\tau)_M\|^2_{L^2(\Omega_l)} \le \frac{\tau}{\omega}\sum_{k=1}^{N-1} \|\mathbf{f}_{k+1}\|^2_{L^2(\Omega_l)} \exp\left(\frac{T}{\omega}\right). \tag{3.12}$$

Together with

$$\lim_{N\to\infty} \tau \sum_{k=1}^{N-1} \|\mathbf{f}_{k+1}\|^2_{L^2(\Omega_l)} = \|\mathbf{f}\|^2_{L^2(\Omega)},$$

(3.12) yields the existence of $0 < K_1 < \infty$, independent of $N \in \mathbb{N}$, such that

$$\|(\mathrm{D}\mathbf{r}^\tau)_M\|^2_{L^2(\Omega_l)} \le K_1. \tag{3.13}$$

Combining (3.11) and (3.13) gives finally the existence of $0 < K_2 < \infty$, independent of $N \in \mathbb{N}$, such that

$$\|(\partial_{ss}\mathbf{r}^\tau)_M\|^2_{L^2(\Omega_l)} \le K_2. \tag{3.14}$$

The inequalities (3.13), (3.14) together with the Poincaré inequality guarantee the existence of the desired $0 < K < \infty$ in (3.10), independent of $N \in \mathbb{N}$. □

Before providing the stability bound for the Lagrange multiplier, we need to establish some bounds on the inverse of the linearized constraint functional.

Lemma 8. *Let $(e'_{k+1})^{-1} \in L(H^1_{0,l}(\Omega_l); H^2_{0,l}(\Omega_l;\mathbb{R}^3))$ be as in Remark 5, $k = 1, ..., N-1$. Then there exists $0 < L < \infty$, independent of $N \in \mathbb{N}$, such that*

$$\|(e'_{k+1})^{-1}[g]\|_{H^2(\Omega_l)} \leq L\, \|g\|_{H^1(\Omega_l)}, \qquad \|(e'_{k+1})^{-1}[g]\|_{L^2(\Omega_l)} \leq L\, \|g\|_{L^2(\Omega_l)},$$
$$\left\| \left(\mathrm{D}\left((e')^{-1}\right)^{\tau}\right)_{k+1}[g] \right\|_{L^2(\Omega_l)} \leq L\, \|g\|_{H^1(\Omega_l)}, \tag{3.15}$$

for all $g \in H^1_{0,l}(\Omega_l)$.

Proof. Let $g \in H^1_{0,l}(\Omega_l)$. As consequence of the Poincaré inequality, Lemma 3 and (3.10), we find constants $0 < L_1, L_2, L_3 < \infty$ (independent of $N \in \mathbb{N}$) such that

$$\begin{aligned}\|(e'_{k+1})^{-1}[g]\|_{H^2(\Omega_l)} &\leq L_1 \left\| \partial_s \left(\frac{g\, \partial_s \mathbf{r}_k}{|\partial_s \mathbf{r}_k|^2} \right) \right\|_{L^2(\Omega_l)} \leq L_1 \left(\|\partial_s g\, \partial_s \mathbf{r}_k\|_{L^2(\Omega_l)} + 3\|g\, \partial_{ss} \mathbf{r}_k\|_{L^2(\Omega_l)} \right) \\ &\leq L_2 \left(\|\partial_s \mathbf{r}_k\|_{C^0(\overline{\Omega_l})} + 3\|\partial_{ss} \mathbf{r}_k\|_{L^2(\Omega_l)} \right) \|g\|_{H^1(\Omega_l)} \leq L_3 \|g\|_{H^1(\Omega_l)}.\end{aligned}$$

Using Cauchy–Schwarz inequality and Lemma 3, we obtain

$$\|(e'_{k+1})^{-1}[g]\|^2_{L^2(\Omega_l)} = \int_0^l \left| \int_u^l \frac{g\, \partial_s \mathbf{r}_k}{2|\partial_s \mathbf{r}_k|^2}\, ds \right|^2 du \leq \frac{3l^2}{4} \int_0^l \frac{g^2 |\partial_s \mathbf{r}_k|^2}{|\partial_s \mathbf{r}_k|^4}\, ds \leq \frac{3l^2}{4} \|g\|^2_{L^2(\Omega_l)}.$$

The estimation of the discrete derivative is a little more lengthly. Integration by parts yields

$$\begin{aligned}\left\| \left(\mathrm{D}\left((e')^{-1}\right)^{\tau}\right)_{k+1}[g] \right\|^2_{L^2(\Omega_l)} &= \int_0^l \left| \int_u^l \frac{g\, \partial_s \mathbf{r}_k}{2\tau |\partial_s \mathbf{r}_k|^2} - \frac{g\, \partial_s \mathbf{r}_{k-1}}{2\tau |\partial_s \mathbf{r}_{k-1}|^2}\, ds \right|^2 du \\
&\leq \int_0^l \left| \int_u^l \frac{g\, \partial_s(\mathbf{r}_k - \mathbf{r}_{k-1})}{\tau |\partial_s \mathbf{r}_k|^2}\, ds \right|^2 du + \int_0^l \left| \int_u^l g\, \partial_s \mathbf{r}_{k-1} \left(\frac{1}{\tau |\partial_s \mathbf{r}_k|^2} - \frac{1}{\tau |\partial_s \mathbf{r}_{k-1}|^2} \right) ds \right|^2 du \\
&\leq 2 \int_0^l \left| \int_u^l \frac{\mathbf{r}_k - \mathbf{r}_{k-1}}{\tau}\, \partial_s \left(\frac{g}{|\partial_s \mathbf{r}_k|^2} \right) ds \right|^2 du + 2 \int_0^l \left| \frac{\mathbf{r}_k - \mathbf{r}_{k-1}}{\tau} \frac{g}{|\partial_s \mathbf{r}_k|^2} \right|^2 ds \\
&\quad + \int_0^l \left| \int_u^l \frac{g\, \partial_s \mathbf{r}_{k-1}}{\tau |\partial_s \mathbf{r}_k|^2 |\partial_s \mathbf{r}_{k-1}|^2}\, \partial_s(\mathbf{r}_k - \mathbf{r}_{k-1}) \cdot \partial_s(\mathbf{r}_k + \mathbf{r}_{k-1})\, ds \right|^2 du \\
&\leq L_4\, \|g\|^2_{H^1(\Omega_l)} + 2 \int_0^l \left| \int_u^l \frac{\mathbf{r}_k - \mathbf{r}_{k-1}}{\tau} \cdot \partial_s \left(\partial_s(\mathbf{r}_k + \mathbf{r}_{k-1}) \frac{g\, \partial_s \mathbf{r}_{k-1}}{|\partial_s \mathbf{r}_k|^2 |\partial_s \mathbf{r}_{k-1}|^2} \right) ds \right|^2 du \\
&\quad + 2 \int_0^l \left| \frac{\mathbf{r}_k - \mathbf{r}_{k-1}}{\tau} \cdot \partial_s(\mathbf{r}_k + \mathbf{r}_{k-1}) \frac{g\, \partial_s \mathbf{r}_{k-1}}{|\partial_s \mathbf{r}_k|^2 |\partial_s \mathbf{r}_{k-1}|^2} \right|^2 ds \\
&\leq L_5\, \|g\|^2_{H^1(\Omega_l)}\end{aligned}$$

for some $0 < L_4, L_5 < \infty$ (independent of $N \in \mathbb{N}$). Here, again, we used Lemma 3 and the estimates provided in Proposition 7 in several steps. □

Proposition 9 (Stability bound for Lagrange multiplier)**.** *Let $\lambda_{k+1} \in H^{-1}(\Omega_l)$ be as in (3.8), $k = 1, ..., N-1$, and let λ^τ be the corresponding linear interpolation. Then $\lambda^\tau \in H^{-1,0}(\Omega)$ and there exists $0 < C < \infty$, independent of $N \in \mathbb{N}$, such that*

$$\left| {}_{H^1_{0,l,T}(\Omega)}\langle g \otimes h, \lambda^\tau \rangle_{H^{-1}(\Omega)} \right| \leq C\, \|g\|_{H^1(\Omega_l)} \|h\|_{H^1(\Omega_T)} \tag{3.16}$$

for all $g \in H^1_{0,l}(\Omega_l)$, $h \in H^1_{0,T}(\Omega_T)$.

Proof. According to (3.8), the Lagrange multiplier is given by

$$\lambda_{k+1} = -J'_{k+1}(\mathbf{r}_{k+1})(e'_{k+1})^{-1}, \quad k = 1, \dots, N-1.$$

Let $g \in H^1_{0,l}(\Omega_l)$ and $h \in H^1_{0,T}(\Omega_T)$. Then, due to the imposed initial and boundary conditions

$$\begin{aligned}
{}_{H^1_{0,l,T}(\Omega)}\langle g \otimes h, \lambda^\tau \rangle_{H^{-1}(\Omega)} &= \int_0^T {}_{H^1_{0,l}(\Omega_l)}\langle g, \lambda^\tau(t)\rangle_{H^{-1}(\Omega_l)} h(t)\, dt \\
&= \sum_{k=1}^{N-1} \int_{t_k}^{t_{k+1}} {}_{H^1_{0,l}(\Omega_l)}\left\langle g, \frac{t-t_k}{\tau}(\lambda_{k+1}-\lambda_k) + \lambda_k \right\rangle_{H^{-1}(\Omega_l)} h(t)\, dt \\
&= \sum_{k=1}^{N-1} {}_{H^1_{0,l}(\Omega_l)}\langle g, \lambda_{k+1}\rangle_{H^{-1}(\Omega_l)} h^{(-1)}_{k+1} - {}_{H^1_{0,l}(\Omega_l)}\langle g, \lambda_k\rangle_{H^{-1}(\Omega_l)} h^{(-1)}_k \\
&\quad - \Big({}_{H^1_{0,l}(\Omega_l)}\langle g, \lambda_{k+1}\rangle_{H^{-1}(\Omega_l)} - {}_{H^1_{0,l}(\Omega_l)}\langle g, \lambda_k\rangle_{H^{-1}(\Omega_l)} \Big) \frac{1}{\tau}\int_{t_k}^{t_{k+1}} h^{(-1)}(t)\, dt \\
&= \tau \sum_{k=2}^{N-1} {}_{H^1_{0,l}(\Omega_l)}\langle g, \lambda_k\rangle_{H^{-1}(\Omega_l)} \big(\mathrm{D}^2 h^{(-2)}\big)_{k+1} - {}_{H^1_{0,l}(\Omega_l)}\langle g, \lambda_N\rangle_{H^{-1}(\Omega_l)} \big(\mathrm{D} h^{(-2)}\big)_N,
\end{aligned} \tag{3.17}$$

where $h^{(-j)}$ is the primitive function of $h^{(-j+1)}$ with $h^{(-j)}(T) = 0$, $j = 1, 2$, $h^{(0)} = h$. Furthermore,

$$\begin{aligned}
{}_{H^1_{0,l}(\Omega_l)}\langle g, \lambda_{k+1}\rangle_{H^{-1}(\Omega_l)} &= -J'_{k+1}(\mathbf{r}_{k+1})(e'_{k+1})^{-1}[g] \\
&= -2\Big(\omega\big((\mathrm{D}^2\mathbf{r}^\tau)_{k+1}, (e'_{k+1})^{-1}[g]\big)_{L^2(\Omega_l)} + b\big(\partial_{ss}\mathbf{r}_{k+1}, \partial_{ss}((e'_{k+1})^{-1}[g])\big)_{L^2(\Omega_l)} \\
&\qquad - \big(\mathbf{f}_{k+1}, (e'_{k+1})^{-1}[g]\big)_{L^2(\Omega_l)} \Big).
\end{aligned} \tag{3.18}$$

Using (3.17) and (3.18), now we estimate ${}_{H^1_{0,l,T}(\Omega)}\langle g \otimes h, \lambda^\tau \rangle_{H^{-1}(\Omega)}$ term by term. First we consider

$$\begin{aligned}
&\left| \tau \sum_{k=2}^{N-1} \left(\frac{\mathbf{r}_k - 2\mathbf{r}_{k-1} - \mathbf{r}_{k-2}}{\tau^2}, (e'_k)^{-1}[g] \right)_{L^2(\Omega_l)} \big(\mathrm{D}^2 h^{(-2)}\big)_{k+1} \right| \\
&\qquad \le \left| \tau \sum_{k=2}^{N-2} \Big((\mathrm{D}\mathbf{r}^\tau)_k, \big(\mathrm{D}\big((e')^{-1}\big)^\tau\big)_{k+1}[g] \Big)_{L^2(\Omega_l)} \big(\mathrm{D}^2 h^{(-2)}\big)_{k+1} \right| \\
&\qquad\quad + \left| \tau \sum_{k=2}^{N-2} \big((\mathrm{D}\mathbf{r}^\tau)_k, (e'_{k+1})^{-1}[g] \big)_{L^2(\Omega_l)} \big(\mathrm{D}^3 h^{(-2)}\big)_{k+2} \right| \\
&\qquad\quad + \left| \big((\mathrm{D}\mathbf{r}^\tau)_{N-1}, (e'_{N-1})^{-1}[g] \big)_{L^2(\Omega_l)} \big(\mathrm{D}^2 h^{(-2)}\big)_N \right| \\
&\qquad \le KL\|g\|_{H^1(\Omega_l)}\, \tau \sum_{k=2}^{N-1} \Big(\Big|\big(\mathrm{D}^2 h^{(-2)}\big)_{k+1}\Big| + \Big|\big(\mathrm{D}^3 h^{(-2)}\big)_{k+1}\Big| \Big) \\
&\qquad \le 5\sqrt{T} KL \|g\|_{H^1(\Omega_l)} \|h\|_{H^1(\Omega_T)}
\end{aligned}$$

where we used several times (3.10) and (3.15). Since $h^{(-1)}(T) = h(T) = 0$, we have

$$\left| \left(\frac{\mathbf{r}_N - 2\mathbf{r}_{N-1} - \mathbf{r}_{N-2}}{\tau^2}, (e'_N)^{-1}[g] \right)_{L^2(\Omega_l)} \big(\mathrm{D} h^{(-2)}\big)_N \right| \le C_1 \|g\|_{H^1(\Omega_l)} \|h\|_{H^1(\Omega_T)}$$

for some $0 < C_1 < \infty$, independent of $N \in \mathbb{N}$. Using (3.10), (3.15) and the continuity of $\mathbf{f}$, a derivation of an appropriate bound for the remaining four terms is straight forward. □

3.4. **Convergence to a weak solution.**

Theorem 10. *There exists a sequence of discretizations* $(\tau_n)_{n\in\mathbb{N}}$, $\mathbf{r} \in H^{2,1}(\Omega)$ *and* $\lambda \in H^{-\beta}(\Omega)$ *such that*

$$\lim_{n\to\infty} \mathbf{r}^{\tau_n} = \mathbf{r} \quad \text{in} \quad C^0([0,T]; L^2(\Omega_l;\mathbb{R}^3)), \tag{3.19a}$$

$$\lim_{n\to\infty} \mathbf{r}^{\tau_n} = \mathbf{r} \quad \text{weakly in} \quad H^{2,1}(\Omega;\mathbb{R}^3), \tag{3.19b}$$

$$\lim_{n\to\infty} \mathbf{r}^{\tau_n} = \mathbf{r} \quad \text{strongly in} \quad L^2(\Omega;\mathbb{R}^3), \tag{3.19c}$$

$$\lim_{n\to\infty} \lambda^{\tau_n} = \lambda \quad \text{strongly in} \quad H^{-\beta}(\Omega), \tag{3.19d}$$

for all $3/2 < \beta < \infty$. *Furthermore,* $(\mathbf{r}, \lambda)$ *are weakly solving* (2.1), *i.e.,*

$$-\omega\, (\partial_t \mathbf{r}, \partial_t \boldsymbol{\phi})_{L^2(\Omega)} = -{}_{H^{1,0}_{0,l,T}(\Omega)}\langle \partial_s \mathbf{r}\cdot\partial_s\boldsymbol{\phi}, \lambda\rangle_{H^{-1,0}(\Omega)} - b\,(\partial_{ss}\mathbf{r}, \partial_{ss}\boldsymbol{\phi})_{L^2(\Omega)} + (\mathbf{f}, \boldsymbol{\phi})_{L^2(\Omega)} \tag{3.20a}$$

$$\left(|\partial_s \mathbf{r}|^2, \partial_t \phi\right)_{L^2(\Omega)} = 0, \tag{3.20b}$$

for all $\boldsymbol{\phi} \in H^3_{0,l}(\Omega_l;\mathbb{R}^3) \otimes H^1_{0,T}(\Omega_T)$ *and all* $\phi \in C_c^\infty(\Omega)$. *Furthermore, for all* $0 \le \gamma < 1/2$

$$\lim_{n\to\infty} \mathbf{r}^{\tau_n}(t) = \mathbf{r}(t) \qquad \text{in} \quad C^{0,\gamma}([0,l];\mathbb{R}^3),$$

$$\lim_{n\to\infty} \partial_s\mathbf{r}^{\tau_n}(t) = \partial_s\mathbf{r}(t) \quad \text{in} \quad C^{0,\gamma}([0,l];\mathbb{R}^3),$$

for all $t \in D$, *where* $D \subset [0,T]$ *is countable (in the following we are choosing* $D \subset [0,T]$ *dense and let* $0, T \in D$*). Moreover,* $\mathbf{r}$ *has a (unique) continuous version (denoted by the same symbol) and*

$$\mathbf{r}(l,t) = \mathbf{0}, \quad \partial_s\mathbf{r}(l,t) = -\mathbf{e_g}, \quad \mathbf{r}(s,0) = (l-s)\mathbf{e_g} \quad \text{for all } (s,t) \in [0,l]\times[0,T],$$

and even

$$|\partial_s \mathbf{r}(s,t)|^2 = 1 \ \text{ for a.e. } \ (s,t) \in [0,l]\times[0,T]. \tag{3.21}$$

Remark 11. *(i) Theorem 10 provides existence of a weak solution to the nonlinear fourth order partial differential system with algebraic constraint given in* (2.1). *Taking into account the regularity of the weak solution, the Dirichlet and Neumann boundary conditions in* (2.2) *are fulfilled as far as possible in this situation.*
(ii) Under the assumption of uniqueness of a weak solution as in Theorem 10, one has convergence even for all discretizations $(\tau_n)_{n\in\mathbb{N}}$.
(iii) The pairing ${}_{H^{1,0}_{0,l,T}(\Omega)}\langle \partial_s \mathbf{r}\cdot\partial_s\boldsymbol{\phi}, \lambda\rangle_{H^{-1,0}(\Omega)}$ *in* (3.20a) *has to be understood in the sense of a uniquely closable bilinear form. I.e., if* $(f_n)_{n\in\mathbb{N}}$ *converges weakly to* f *in* $H^{1,0}_{0,l,T}(\Omega)$, $(F_n)_{n\in\mathbb{N}}$ *is a sequence in* $H^{-1,0}(\Omega)$ *which converges strongly to* F *in* $H^{-2}(\Omega)$, $\lim_{n\to\infty} {}_{H^{1,0}_{0,l,T}(\Omega)}\langle f_n, F_n\rangle_{H^{-1,0}(\Omega)}$ *exists and* $f = 0$ *or* $F = 0$, *then* $\lim_{n\to\infty} {}_{H^{1,0}_{0,l,T}(\Omega)}\langle f_n, F_n\rangle_{H^{-1,0}(\Omega)} = 0$.

Proof. From the first two estimates in (3.10) we can conclude that $\mathbf{r}^\tau$ is uniformly (in $N \in \mathbb{N}$) Lipschitz continuous in $C^0([0,T]; L^2(\Omega_l;\mathbb{R}^3))$ and

$$\{\mathbf{r}^\tau(t) \,|\, t \in [0,T]\} \subset \{\mathbf{v} \in H^2(\Omega_l;\mathbb{R}^3) \,|\, \|\partial_{ss}\mathbf{v}\|_{L^2(\Omega_l)} \le K\},$$

which is a relative compact subset of $L^2(\Omega_l;\mathbb{R}^3)$. Thus, there exists sequence of discretizations $(\tau_n)_{n\in\mathbb{N}}$ and $\mathbf{r} \in C^0([0,T]; L^2(\Omega_l;\mathbb{R}^3))$ such that $\mathbf{r}^{\tau_n}$ converges to $\mathbf{r}$ in $C^0([0,T]; L^2(\Omega_l;\mathbb{R}^3))$ for $n \to \infty$.

The third estimate in (3.10) gives the existence of a subsequence $(\tau_n)_{n\in\mathbb{N}}$ (denoted the same) and $\tilde{\mathbf{r}} \in H^{2,1}(\Omega;\mathbb{R}^3)$ such that $\mathbf{r}^{\tau_n}$ converges weakly to $\tilde{\mathbf{r}}$ in $H^{2,1}(\Omega;\mathbb{R}^3)$ for $n \to \infty$. Since convergence in $C^0([0,T]; L^2(\Omega_l;\mathbb{R}^3))$ implies strong convergence in $L^2(\Omega;\mathbb{R}^3)$ as well as weak convergence in $H^{2,1}(\Omega;\mathbb{R}^3)$ implies weak convergence in $L^2(\Omega;\mathbb{R}^3)$, we have $\tilde{\mathbf{r}} = \mathbf{r}$. In particular, this shows (3.19a)-(3.19c).

From (3.16) together with the fact that the embedding $H^{\beta_1}(\Omega_a) \subset H^1(\Omega_a)$, $a \in \{l, T\}$, is Hilbert–Schmidt for all $3/2 < \beta_1 < \infty$, we obtain by the kernel theorem, see e.g. [4, Chap. 1,§2.3], that λ^τ is uniformly (in $N \in \mathbb{N}$) bounded in $H^{-\beta_1}(\Omega)$ for all $3/2 < \beta_1 < \infty$. Since the embedding

$H^{-\beta_1}(\Omega) \subset H^{-\beta_1-\beta_2}(\Omega)$ is compact for all $0 < \beta_2 < \infty$, there exists a subsequence $(\tau_n)_{n\in\mathbb{N}}$ and $\lambda \in H^{-\beta}(\Omega)$ such that λ^{τ_n} converges strongly to λ in $H^{-\beta}(\Omega)$ as $n \to \infty$ for all $3/2 < \beta < \infty$.

Multiplying the linear interpolation of (3.9a) with a time-dependent test function and integrating w.r.t. time yields

$$\begin{aligned}\omega\left(\left(\mathrm{D}^2\mathbf{r}^\tau\right)^\tau, \boldsymbol{\phi}\right)_{L^2(\Omega)} &= -{}_{H^1_{0,l,T}(\Omega)}\langle \partial_s\mathbf{r}^\tau(\cdot-\tau)\cdot\partial_s\boldsymbol{\phi}, \lambda^\tau\rangle_{H^{-1}(\Omega)} \\ &\quad - b\left(\partial_{ss}\mathbf{r}^\tau, \partial_{ss}\boldsymbol{\phi}\right)_{L^2(\Omega)} + (\mathbf{f}^\tau, \boldsymbol{\phi})_{L^2(\Omega)},\end{aligned} \tag{3.22}$$

for all $\boldsymbol{\phi} \in H^2_{0,l}(\Omega_l;\mathbb{R}^3) \otimes H^1_{0,T}(\Omega_T)$ (because $\lambda_0 = \lambda_1 = 0$, on $[-\tau, 0]$ we can assign to the function $\partial_s\mathbf{r}^\tau(\cdot-\tau)$ any value, for simplicity we choose zero). Since $\partial_{ss}\mathbf{r}^{\tau_n}$ converges weakly to $\partial_{ss}\mathbf{r}$ and $\mathbf{f}^{\tau_n}$ converges strongly to $\mathbf{f}$, both in $L^2(\Omega;\mathbb{R}^3)$, as $n\to\infty$, we have

$$\lim_{n\to\infty}(\partial_{ss}\mathbf{r}^{\tau_n}, \partial_{ss}\boldsymbol{\phi})_{L^2(\Omega)} = (\partial_{ss}\mathbf{r}, \partial_{ss}\boldsymbol{\phi})_{L^2(\Omega)}, \quad \lim_{n\to\infty}(\mathbf{f}^{\tau_n}, \boldsymbol{\phi})_{L^2(\Omega)} = (\mathbf{f}, \boldsymbol{\phi})_{L^2(\Omega)}, \tag{3.23}$$

for all $\boldsymbol{\phi} \in H^2_{0,l}(\Omega_l;\mathbb{R}^3) \otimes H^1_{0,T}(\Omega_T)$. Furthermore, integration by parts yields

$$\begin{aligned}\left(\left(\mathrm{D}^2\mathbf{r}^\tau\right)^\tau, \boldsymbol{\phi}\right)_{L^2(\Omega)} &= \sum_{k=0}^{N-1}\int_{t_k}^{t_{k+1}}\int_0^l \left(\frac{t-t_k}{\tau}\Big((\mathrm{D}^2\mathbf{r}^\tau)_{k+1} - (\mathrm{D}^2\mathbf{r}^\tau)_k\Big) + (\mathrm{D}^2\mathbf{r}^\tau)_k\right)\cdot\boldsymbol{\phi}\,ds\,dt \\
&= \sum_{k=0}^{N-1}\int_{t_k}^{t_{k+1}}\int_0^l \frac{1}{\tau}\left(\left(\frac{t-t_k}{\tau}\Big((\mathrm{D}\mathbf{r}^\tau)_{k+1} - (\mathrm{D}\mathbf{r}^\tau)_k\Big) + (\mathrm{D}\mathbf{r}^\tau)_k\right)\right. \\
&\quad \left. - \left(\frac{t-t_k}{\tau}\Big((\mathrm{D}\mathbf{r}^\tau)_k - (\mathrm{D}\mathbf{r}^\tau)_{k-1}\Big) + (\mathrm{D}\mathbf{r}^\tau)_{k-1}\right)\right)\cdot\boldsymbol{\phi}\,ds\,dt \\
&= -\sum_{k=0}^{N-2}\int_{t_k}^{t_{k+1}}\int_0^l \left(\frac{t-t_k}{\tau}\Big((\mathrm{D}\mathbf{r}^\tau)_{k+1} - (\mathrm{D}\mathbf{r}^\tau)_k\Big) + (\mathrm{D}\mathbf{r}^\tau)_k\right)\cdot\frac{\boldsymbol{\phi}(\cdot+\tau)-\boldsymbol{\phi}}{\tau}\,ds\,dt \\
&\quad + \int_{t_{N-1}}^{t_N}\int_0^l \frac{1}{\tau}\left(\frac{t-t_{N-1}}{\tau}\Big((\mathrm{D}\mathbf{r}^\tau)_N - (\mathrm{D}\mathbf{r}^\tau)_{N-1}\Big) + (\mathrm{D}\mathbf{r}^\tau)_{N-1}\right)\cdot\boldsymbol{\phi}\,ds\,dt \\
&= \int_0^l \Big((\mathrm{D}\mathbf{r}^\tau)_N - (\mathrm{D}\mathbf{r}^\tau)_{N-1}\Big)\cdot\frac{1}{\tau^2}\int_{t_{N-1}}^{t_N}\int_\cdot^{t_N}\boldsymbol{\phi}\,du\,dt\,ds + \int_0^l (\mathrm{D}\mathbf{r}^\tau)_{N-1}\cdot\frac{1}{\tau}\int_{t_{N-1}}^{t_N}\boldsymbol{\phi}\,dt\,ds \\
&\quad - \int_{t_1}^{t_{N-1}}\int_0^l \left(\mathrm{D}\mathbf{r}^\tau\right)^\tau\cdot\frac{\boldsymbol{\phi}(\cdot+\tau)-\boldsymbol{\phi}}{\tau}\,ds\,dt.\end{aligned} \tag{3.24}$$

By (3.10) together with the boundary conditions imposed on $\boldsymbol{\phi}$ now from (3.24) it follows

$$\lim_{n\to\infty}\left(\left(\mathrm{D}^2\mathbf{r}^{\tau_n}\right)^{\tau_n}, \boldsymbol{\phi}\right)_{L^2(\Omega)} = -\lim_{n\to\infty}\int_{t_1}^{t_{N-1}}\int_0^l \left(\mathrm{D}\mathbf{r}^{\tau_n}\right)^{\tau_n}\cdot\frac{\boldsymbol{\phi}(\cdot+\tau_n)-\boldsymbol{\phi}}{\tau_n}\,ds\,dt = -\left(\partial_t\mathbf{r}, \partial_t\boldsymbol{\phi}\right)_{L^2(\Omega)}, \tag{3.25}$$

where in the last step we used that $\partial_t\mathbf{r}^{\tau_n}$ converges weakly to $\partial_t\mathbf{r}$, $(\boldsymbol{\phi}(\cdot+\tau_n)-\boldsymbol{\phi})/\tau_n$ converges strongly to $\partial_t\boldsymbol{\phi}$ and $\partial_t\mathbf{r}^{\tau_n} - \left(\mathrm{D}\mathbf{r}^{\tau_n}\right)^{\tau_n}$ converges strongly to 0, all in $L^2(\Omega;\mathbb{R}^3)$ as $n\to\infty$.

Combining (3.22), (3.23), (3.25) we obtain

$$\begin{aligned}\lim_{n\to\infty}{}_{H^{1,0}_{0,l,T}(\Omega)}\langle\partial_s\mathbf{r}^{\tau_n}(\cdot-\tau_n)\cdot\partial_s\boldsymbol{\phi}, \lambda^{\tau_n}\rangle_{H^{-1,0}(\Omega)} &= \omega\left(\partial_t\mathbf{r}, \boldsymbol{\phi}\right)_{L^2(\Omega)} - b\left(\partial_{ss}\mathbf{r}, \partial_{ss}\boldsymbol{\phi}\right)_{L^2(\Omega)} \\ &\quad + (\mathbf{f}, \boldsymbol{\phi})_{L^2(\Omega)},\end{aligned} \tag{3.26}$$

for all $\boldsymbol{\phi} \in H^2_{0,l}(\Omega_l;\mathbb{R}^3) \otimes H^1_{0,T}(\Omega_T)$. Now we restrict ourself to $\boldsymbol{\phi} \in H^3_{0,l}(\Omega_l;\mathbb{R}^3) \otimes H^1_{0,T}(\Omega_T)$. Since $\partial_s\mathbf{r}^{\tau_n}$ converges weakly to $\partial_s\mathbf{r}$ in $H^{1,0}_{0,l,T}(\Omega;\mathbb{R}^3)$ and $\partial_s\boldsymbol{\phi}, \partial_{ss}\boldsymbol{\phi}$ are bounded functions, also $\partial_s\mathbf{r}^{\tau_n}(\cdot-\tau_n)\cdot\partial_s\boldsymbol{\phi}$ converges weakly to $\partial_s\mathbf{r}\cdot\partial_s\boldsymbol{\phi}$ in $H^{1,0}_{0,l,T}(\Omega)$ as $n\to\infty$. Furthermore, λ^{τ_n} converges strongly to λ, e.g. in $H^{-2}(\Omega)$, as $n\to\infty$. Thus

$$\lim_{n\to\infty}{}_{H^{1,0}_{0,l,T}(\Omega)}\langle\partial_s\mathbf{r}^{\tau_n}(\cdot-\tau_n)\cdot\partial_s\boldsymbol{\phi}, \lambda^{\tau_n}\rangle_{H^{-1,0}(\Omega)} = {}_{H^{1,0}_{0,l,T}(\Omega)}\langle\partial_s\mathbf{r}\cdot\partial_s\boldsymbol{\phi}, \lambda\rangle_{H^{-1,0}(\Omega)}, \tag{3.27}$$

for all $\boldsymbol{\phi} \in H^3_{0,l}(\Omega_l;\mathbb{R}^3) \otimes H^1_{0,T}(\Omega_T)$ in the sense of a uniquely closable bilinear form, see Remark 11(iii). Hence, (3.20a) follows from (3.26) together with (3.27).

Now, using (3.9b), we obtain for all $\phi \in C_c^\infty(\Omega)$

$$\begin{aligned}
\left|\left(|\partial_s \mathbf{r}^\tau|^2, \partial_t \phi\right)_{L^2(\Omega)}\right| &= \left|\sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} \int_0^l \left(\frac{t-t_k}{\tau}\left(\partial_s \mathbf{r}_{k+1} - \partial_s \mathbf{r}_k\right) + \partial_s \mathbf{r}_k\right)^2 \partial_t \phi \, ds \, dt\right| \\
&= \left|\frac{2}{\tau}\sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} \int_0^l \left(\frac{t-t_k}{\tau}\left(\partial_s \mathbf{r}_{k+1} - \partial_s \mathbf{r}_k\right) + \partial_s \mathbf{r}_k\right)\left(\partial_s \mathbf{r}_{k+1} - \partial_s \mathbf{r}_k\right) \phi \, ds \, dt\right| \\
&= 2\sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} \int_0^l \frac{t-t_k}{\tau^2}\left(|\partial_s \mathbf{r}_{k+1}|^2 - |\partial_s \mathbf{r}_k|^2\right)|\phi| \, ds \, dt \\
&\leq 2l \sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} \frac{t-t_k}{\tau^2}(\tau^4 + \tau^3 + \tau^2) \, dt \|\phi\|_{C^0(\overline{\Omega})} \qquad (3.28)
\end{aligned}$$

due to Lemma 3. Because $\mathbf{r}^{\tau_n}$ converges strongly to $\mathbf{r}$ and $\partial_s \mathbf{r}^{\tau_n}$, $\partial_{ss} \mathbf{r}^{\tau_n}$ converge weakly to $\partial_s \mathbf{r}$, $\partial_{ss} \mathbf{r}$, respectively, in $L^2(\Omega;\mathbb{R}^3)$ as $n \to \infty$, by an integration by parts together with (3.28) we can conclude

$$\begin{aligned}
\left(|\partial_s \mathbf{r}|^2, \partial_t \phi\right)_{L^2(\Omega)} &= -\left(\mathbf{r} \cdot \partial_s \mathbf{r}, \partial_{st} \phi\right)_{L^2(\Omega)} - \left(\mathbf{r} \cdot \partial_{ss} \mathbf{r}, \partial_t \phi\right)_{L^2(\Omega)} \\
&= -\lim_{n\to\infty} \left(\partial_{st}\phi \; \mathbf{r}^{\tau_n}, \partial_s \mathbf{r}^{\tau_n}\right)_{L^2(\Omega)} - \lim_{n\to\infty} \left(\partial_t \phi \; \mathbf{r}^{\tau_n}, \partial_{ss} \mathbf{r}^{\tau_n}\right)_{L^2(\Omega)} \\
&= \lim_{n\to\infty} \left(|\partial_s \mathbf{r}^{\tau_n}|^2, \partial_t \phi\right)_{L^2(\Omega)} = 0 \qquad (3.29)
\end{aligned}$$

for all $\phi \in C_c^\infty(\Omega)$, i.e., (3.20b) is shown.

Due to the second estimate in (3.10), for each $t \in [0,T]$ there exists a subsequence $(\tau_n)_{n\in\mathbb{N}}$ (depending on t) such that

$$\lim_{n\to\infty} \mathbf{r}^{\tau_n}(t) = \mathbf{r}(t), \quad \lim_{n\to\infty} \partial_s \mathbf{r}^{\tau_n}(t) = \partial_s \mathbf{r}(t) \quad \text{both in } C^{0,\gamma}([0,l];\mathbb{R}^3), \qquad (3.30)$$

for all $0 \leq \gamma < 1/2$. Let $D \subset [0,T]$ be countable. Then, by dropping to subsequences and taking the diagonal sequence, we obtain (3.30) for all $t \in D$. Here we choose $D \subset [0,T]$ dense with $0, T \in [0,T]$. From this, together with the first and second estimate in (3.10), we can conclude that $\mathbf{r}$ has a (unique) continuous version on $[0,l]\times[0,T]$ (which we denote by the same symbol). Moreover,

$$\mathbf{r}(l,t) = \mathbf{0}, \quad \partial_s \mathbf{r}(l,t) = -\mathbf{e_g}, \quad \mathbf{r}(s,0) = (l-s)\mathbf{e_g} \quad \text{for all } (s,t) \in [0,l]\times[0,T]. \qquad (3.31)$$

Finally, (3.29) together with (3.31) implies (3.21). □

4. Numerical study

On every time level t_k the semi-discretized fiber system (3.1) corresponds to a constrained minimization problem (3.6) in a Hilbert space setting. In this section we solve the optimization problem numerically in a finite dimensional approximation space (finite element space) by applying a projected gradient method with Armijo step size rule. The convergence of the method is ensured by the choice of a minimal projection. The temporal evolution of the fiber behavior is then realized by a recursive solving of the optimization problems where the solution $\mathbf{r}_k$ is used as initial guess in the projected gradient method for t_{k+1}. The numerical results coincide well with the previous theoretical investigations.

4.1. **Projected gradient method on finite element spaces.** Based on the classical method of steepest descent the projected gradient method is of first order convergence. But the theoretical convergence result requires the choice of a minimal projection. However, in practice also other projection mappings might lead to satisfying results; for details see e.g. [9]. For the fiber system in the Hilbert space setting the projected gradient method is given by Algorithm 12. Note that here and in the following we suppress the time index $_k$ to facilitate the readability.

Algorithm 12 (Projected gradient method)**.**
Let $P : V \to K$ be a minimal projection mapping onto the set K.
Define $p : V \to V$ by $p(\mathbf{v}) := \mathbf{v} - P(\mathbf{v} - \nabla J(\mathbf{v}))$ via the cost functional J.
Initialize iteration counter $i = 0$, choose $\mathbf{r}^{(0)} \in K$, set absolute and relative tolerances tol_a, tol_r.
While the stationarity measure satisfies $\|p(\mathbf{r}^{(i)})\|_{H^2(\Omega_l)} > tol_a + tol_r\|p(\mathbf{r}^{(0)})\|_{H^2(\Omega_l)}$, do:
- *calculate $\sigma^{(i)}$ by the projected Armijo rule* [9] *such that*

$$J(P(\mathbf{r}_\sigma^{(i)})) < J(\mathbf{r}^{(i)}), \qquad \mathbf{r}_\sigma^{(i)} = \mathbf{r}^{(i)} - \sigma^{(i)} \nabla J(\mathbf{r}^{(i)})$$

- *set $\mathbf{r}^{(i+1)} = P(\mathbf{r}_\sigma^{(i)})$ and update i*

end

For the numerical realization we apply the projected gradient method to finite dimensional approximation spaces. Therefore we introduce finite element spaces $H_h \subset H^2(\Omega_l)$ and $V_h \subset V$. Depending on the definition of K_h we propose and investigate different projections P_h. We indicate all quantities associated to the finite element discretization by the space index $_h$ that should not be mistaken for the time index $_k$.

Considering the space interval $[0, l]$, we introduce an equidistant grid $\{s_j = (j-1)h \,|\, j = 1, ..., M\}$ with space step size $h = l/(M-1)$. In addition, to simplify the notation we define $s_0 = s_1$ and $s_{M+1} = s_M$. Certainly one can also think of an irregular grid with different h_j, then the partition is assumed to satisfy $h = \max_j h_j \to 0$ as $M \to \infty$. As approximation space $H_h \subset H^2(\Omega_l)$ we choose the finite element space of piecewise cubic polynomials for function and first derivative. In particular

$$\psi_j(s) = \begin{cases} (2 + 3\xi - \xi^3)/4, & \xi = (2s - (s_j + s_{j-1}))/h, \quad s \in [s_{j-1}, s_j] \\ (2 - 3\xi + \xi^3)/4, & \xi = (2s - (s_{j+1} + s_j))/h, \quad s \in [s_j, s_{j+1}] \\ 0 & \qquad\qquad\qquad\qquad\qquad\qquad\quad else \end{cases}$$

$$\phi_j(s) = \begin{cases} h(-1 - \xi + \xi^2 + \xi^3)/8, & \xi = (2s - (s_j + s_{j-1}))/h, \quad s \in [s_{j-1}, s_j] \\ h(\ 1 - \xi - \xi^2 + \xi^3)/8, & \xi = (2s - (s_{j+1} + s_j))/h, \quad s \in [s_j, s_{j+1}] \\ 0 & \qquad\qquad\qquad\qquad\qquad\qquad\quad else \end{cases}$$

$j = 1, ..., M$ form a node basis, i.e.

$$\psi_j(s_i) = \delta_{ij}, \quad \partial_s \psi_j(s_i) = 0, \quad \phi_j(s_i) = 0, \quad \partial_s \phi_j(s_i) = \delta_{ij} \quad \text{for all } i, j = 1, ..., M.$$

Then any function $\mathbf{v}_h \in H_h \subset H^2(\Omega_l; \mathbb{R}^n)$ can be represented by

$$\mathbf{v}_h(s) = \sum_{j=1}^{M} \mathsf{v}_j \psi_j(s) + \mathsf{v}'_j \phi_j(s) \tag{4.1}$$

via its coefficient tuple $\mathsf{v} = (\mathsf{v}_1, ...\mathsf{v}_M, \mathsf{v}'_1, ...\mathsf{v}'_M)^T \in \mathbb{R}^{2Mn}$ with $\mathsf{v}_j, \mathsf{v}'_j \in \mathbb{R}^n$. In the finite dimensional fiber space $V_h \subset H_h$, the degrees of freedom reduce by $2n$ due to the Dirichlet boundary conditions given for $s = l$ that fix the coefficients v_M and v'_M. Identifying the function $\mathbf{v}_h$ with its coefficient tuple v, the cost functional of (3.4) can be expressed as quadratic function in terms of the coefficients,

$$J_h(\mathsf{v}) = \mathsf{v}^T \mathsf{A} \mathsf{v} + \mathsf{b}^T \mathsf{v} + c$$
$$\mathsf{A} = \frac{\omega}{\tau^2}\Upsilon + b\Upsilon'', \quad \mathsf{b} = 2\Upsilon\left(\frac{\omega}{\tau^2}\bar{\mathsf{r}} - \mathsf{f}\right), \quad c = \frac{\omega}{\tau^2}\bar{\mathsf{r}}^T \Upsilon \bar{\mathsf{r}}, \quad \bar{\mathsf{r}} = -2\mathsf{r}_k + \mathsf{r}_{k-1}.$$

The sparse symmetric matrices Υ and Υ'' have a block structure of 2×2,

$$\Upsilon = \begin{pmatrix} \Psi\Psi & \Psi\Phi \\ (\Psi\Phi)^T & \Phi\Phi \end{pmatrix}, \qquad \Upsilon'' = \begin{pmatrix} \Psi\Psi'' & \Psi\Phi'' \\ (\Psi\Phi'')^T & \Phi\Phi'' \end{pmatrix},$$

where each block itself consists of $M \times M$ subblocks of size $n \times n$, e.g.

$$(\Psi\Phi)_{ij} = \int_0^l \psi_i \, \phi_j \, ds \; \mathsf{I}, \quad (\Psi\Phi'')_{ij} = \int_0^l \partial_{ss}\psi_i \, \partial_{ss}\phi_j \, ds \; \mathsf{I} \in \mathbb{R}^{n \times n}, \quad i, j = 1, ..., M.$$

With $\mathsf{I} \in \mathbb{R}^{n\times n}$ identity matrix, the subblocks are diagonal matrices, but they only contain non-zero entries for $|i-j| \leq 1$. The notation holds for all other blocks analogously. The time dependence enters J_h via the temporally earlier fiber positions $\bar{\mathsf{r}}$ and the outer forces f.

The linear constraint $e(\mathbf{v}) = \partial_s(\mathbf{v}-\mathbf{r}_k)\cdot\partial_s\mathbf{r}_k = 0$ for $s \in \Omega_l$ (3.3) yields infinitely many independent conditions for the finite number of unknowns in the approximation space. To allow for more than the trivial solution $\mathbf{v}_h = (\mathbf{r}_k)_h$, we must weaken the constraint and pose the condition only on d space points, $d < 2(M-1)n$, that should be equidistantly distributed in Ω_l. The choice of d is a compromise between approximation quality and numerical realization; for big d the set K_h shrinks to a single function that obviously fulfills the constraint on whole Ω_l but does not allow for a dynamic fiber motion $(\mathbf{r}_k)_h = (\mathbf{r}_0)_h$ for all k. For small d we obtain flexibility but pay the prize of a lower approximation quality. Prescribing the constraint at the spatial grid points ($d = M-1$) is a first intuitive choice. It implies the following definition for K_h:

$$K_h = \{\mathbf{v}_h \in V_h \,|\, (\mathsf{v}'_j - \mathsf{r}'_j)\cdot\mathsf{r}'_j = 0,\ j = 1,...,M-1\}. \tag{4.2}$$

where r represents the coefficient tuple to $(\mathbf{r}_k)_h$.

Remark 13. *Alternatively to (4.2), one can certainly use the finite element functions ψ_j, ϕ_j to evaluate the linear constraint not only at the grid points s_j, but also at interior cell points. Imposing the linear constraint for example additionally on the cell midpoints $s_j + h/2$ for $j = 1,...,M-1$, we obtain $d = 2(M-1)$ and*

$$\begin{aligned} K_h = \{\mathbf{v}_h \in V_h \,|\, (\mathsf{v}'_j - \mathsf{r}'_j)\cdot\mathsf{r}'_j = 0,\ \ (\tilde{\mathsf{v}}_{j+1/2} - \tilde{\mathsf{r}}_{j+1/2})\cdot\tilde{\mathsf{r}}_{j+1/2} = 0,\ \ i = 1,...,M-1 \\ \textit{where } \tilde{\mathsf{v}}_{j+1/2} := \frac{3}{2h}(\mathsf{v}_{j+1} - \mathsf{v}_j) - \frac{1}{4}(\mathsf{v}'_{j+1} + \mathsf{v}'_j) \textit{ and } \tilde{\mathsf{r}}_{j+1/2} \textit{ analogously}\}. \end{aligned} \tag{4.3}$$

The restriction $|\partial_s\mathbf{r}_k - \partial_s\mathbf{v}| \leq \tau^2$ on the continuous solution in K (3.6) (to which we refer as τ-inequality) is implicitly contained in the set K_h by an appropriate choice of d.

The determination of the H^2-minimal projection mapping $P_h : V_h \to K_h$ reduces to the solving of a linear system of equations. According to (4.2) the coefficient tuples $\mathsf{v}'_j \in \mathbb{R}^n$ of $\mathbf{v}_h \in K_h$ satisfy $\mathsf{v}'_j = \mathsf{r}'_j + C(\mathsf{r}'_j)$ with $C(\mathsf{r}'_j)$ orthogonal complement to r'_j. Let $\{\mathsf{z}_{j,l}\}$, $l = 1,...,n-1$ be basis of $C(\mathsf{r}'_j)$ and set $\mathsf{v}'_j = \mathsf{r}'_j + \sum_{l=1}^{n-1}\alpha_{j,l}\mathsf{z}_{j,l}$ with unknowns $\alpha_j \in \mathbb{R}^{n-1}$. Then, the projection

$$P_h(\mathbf{y}_h) = \operatorname{argmin}_{\mathbf{v}_h\in K_h}|\mathbf{y}_h - \mathbf{v}_h|^2_{H^2(\Omega_l)}$$

is uniquely determined by the coefficient vector $\mathsf{w} = (\mathsf{v}_1,...,\mathsf{v}_M,\alpha_1,....,\alpha_M)^T \in \mathbb{R}^{(2n-1)M}$ being the solution of the linear system

$$\begin{aligned} \mathsf{P}\mathsf{w} = \mathsf{q}, \quad \mathsf{P} &= \begin{pmatrix} \Psi\Psi'' & \Psi\Phi''\,\mathsf{Z} \\ (\Psi\Phi''\,\mathsf{Z})^T & \mathsf{Z}^T\,\Phi\Phi''\,\mathsf{Z} \end{pmatrix}, \\ \mathsf{q} &= \begin{pmatrix} \Psi\Psi'' & \Psi\Phi'' \\ (\Psi\Phi''\,\mathsf{Z})^T & \mathsf{Z}^T\,\Phi\Phi'' \end{pmatrix}(\mathsf{y}_1,...,\mathsf{y}_M = 0, \mathsf{y}'_1 - \mathsf{r}'_1,...,\mathsf{y}'_M - \mathsf{r}'_M = 0)^T \end{aligned} \tag{4.4}$$

with $\mathsf{v}_M = 0$ and $\alpha_M = 0$ known due to the Dirichlet boundary conditions. The basis vectors to the orthogonal complements are collected in a $n\times M(n-1)$-matrix that is blown up to $\mathsf{Z} \in \mathbb{R}^{Mn\times M(n-1)}$ by M-times row-wise replication. Although, the symmetric block matrix P has to be assembled and decomposed only once in the projected gradient method for every time step t_k, the bottle-neck in the computation lies in the determination of the $\mathsf{z}_{j,l}$. It slows down the performance drastically. The H^2-seminorm induces a vector norm in V_h. Since in finite dimensions all norms are equivalent, one might think of using instead of (4.4) the minimal projection with respect to the Euclidian vector norm. The orthogonal projection is comparably easy and fast to compute:

$$P_{h,2}(\mathbf{y}_h) = \mathbf{v}_h, \quad \text{with} \quad \mathsf{v}_i = \mathsf{y}_i, \quad \mathsf{v}'_i = \mathsf{r}'_i + \mathsf{y}'_i - \frac{\mathsf{y}'_i\cdot\mathsf{r}'_i}{|\mathsf{r}'_i|^2}\mathsf{r}'_i, \quad i = 1,...,M. \tag{4.5}$$

The Dirichlet boundary conditions are consistently fulfilled. For other definitions of K_h (Remark 13) corresponding projection mappings can be formulated straightforward (see [7] for (4.3)).

The numerical realization is performed with MATLAB Version 7.0.1 on a Intel Core 2 processor, 2GHz. The results are scaled with SI-units to allow for a dimensionless presentation.

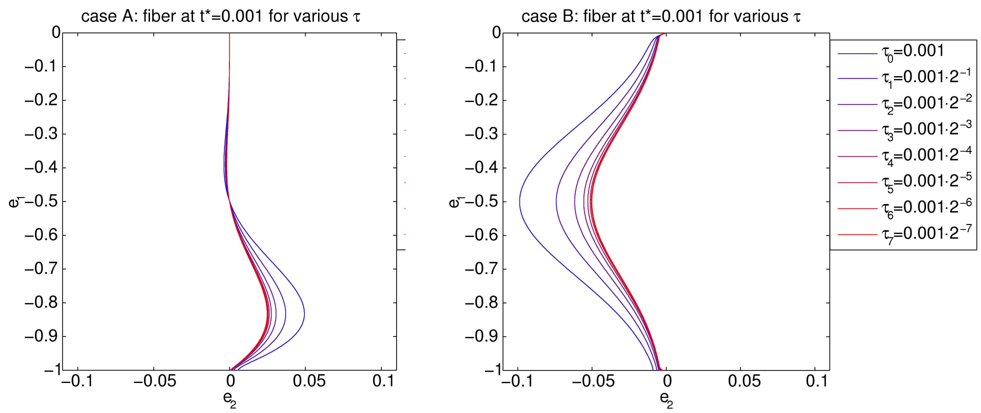

FIGURE 4.2. Fiber position at $t^\star$ computed with different time steps $\tau_i = 2^{-i} \cdot 10^{-3}$, $i = 0, 1, ..., 7$. Acting force in case A (left): $f(s) = (s-1)^3 \sin(2\pi(1-s))$ and in case B (right): $f(s) = -\exp(-10(s-0.5)^2)$.

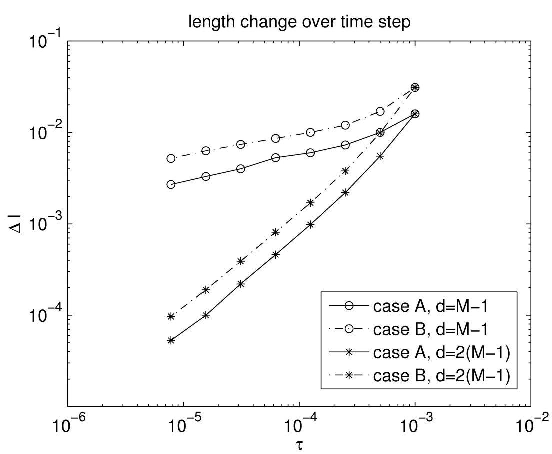

FIGURE 4.3. Length change Δl in dependence of time step for different sets K_h: (4.2) with $d = M-1$ is marked by $\circ$ and (4.3) with $d = 2(M-1)$ by $\star$. The cases A (solid line) and B (dashed-dotted line) refer to the different applied outer forces, cf. Figure 4.2 and Table 4.1.

4.2. **Results and discussion.** The numerical studies show the expected performance of the projected gradient method: the strongest decay of the cost functional J_h happens within the first iterations, then it slows down until the tolerances of the break-up criterion are finally fulfilled. Thereby, the projections P_h (4.4) and $P_{h,2}$ (4.5) yield similar results: the deviation of the minimized function values of J_h is of order $\mathcal{O}(10^{-10})$. Requiring the same number of iterations, the explicit projection scheme with $P_{h,2}$ is much faster than the implicit one with P_h. As mentioned, the bottle-neck of the computation is the assembly of the projection matrix P. Being interested in the temporal evolution of the fiber and hence in an efficient computation of thousands of time levels via the projected gradient method we apply $P_{h,2}$ in the following simulations.

We consider a set-up with $\mathbf{e_g} = -\mathbf{e_1}$ and $\mathbf{f} = f\mathbf{e_2}$ for different exemplary force functions f. The fiber parameters are $l = 1$, $\omega = 10^{-5}$ and $b = 10^{-9}$. The space discretization is chosen to be $M = 300$. The projected gradient method is initialized with the solution of the previous time

case A: $f(s) = (s-1)^3 \sin(2\pi(1-s))$

	efficiency			accuracy		Δl (4.3)
	# iterations ($\varnothing$	Σ)	CPU-time [s]	$\lvert(\mathrm{r}^{\tau_i} - \mathrm{r}^{\tau_7})(t^\star)\rvert$	Δl	
$\boldsymbol{\tau_0}$	1371.0	1371	86	$\mathbf{2.0 \cdot 10^{-1}}$	$1.6 \cdot 10^{-2}$	$1.6 \cdot 10^{-2}$
τ_1	342.0	684	40	$1.0 \cdot 10^{-1}$	$1.0 \cdot 10^{-2}$	$5.5 \cdot 10^{-3}$
τ_2	84.3	337	19	$4.7 \cdot 10^{-2}$	$7.3 \cdot 10^{-3}$	$2.2 \cdot 10^{-3}$
$\boldsymbol{\tau_3}$	21.0	168	11	$\mathbf{2.2 \cdot 10^{-2}}$	$6.0 \cdot 10^{-3}$	$9.8 \cdot 10^{-4}$
τ_4	3.3	52	7	$1.1 \cdot 10^{-2}$	$5.3 \cdot 10^{-3}$	$4.6 \cdot 10^{-4}$
τ_5	3.6	116	14	$5.3 \cdot 10^{-3}$	$4.0 \cdot 10^{-3}$	$2.2 \cdot 10^{-4}$
$\boldsymbol{\tau_6}$	3.8	244	34	$\mathbf{2.1 \cdot 10^{-3}}$	$3.3 \cdot 10^{-3}$	$1.0 \cdot 10^{-4}$
τ_7	4.2	542	80	0	$2.7 \cdot 10^{-3}$	$5.3 \cdot 10^{-5}$

case B: $f(s) = -\exp(-10(s-0.5)^2)$

	efficiency			accuracy		Δl (4.3)
	# iterations ($\varnothing$	Σ)	CPU-time [s]	$\lvert(\mathrm{r}^{\tau_i} - \mathrm{r}^{\tau_7})(t^\star)\rvert$	Δl	
$\boldsymbol{\tau_0}$	1374.0	1374	88	$\mathbf{5.3 \cdot 10^{-1}}$	$3.1 \cdot 10^{-2}$	$3.1 \cdot 10^{-2}$
τ_1	356.0	712	41	$2.6 \cdot 10^{-1}$	$1.7 \cdot 10^{-2}$	$9.9 \cdot 10^{-3}$
τ_2	94.0	376	21	$1.3 \cdot 10^{-1}$	$1.2 \cdot 10^{-2}$	$3.8 \cdot 10^{-3}$
$\boldsymbol{\tau_3}$	24.0	192	12	$\mathbf{5.5 \cdot 10^{-2}}$	$1.0 \cdot 10^{-2}$	$1.7 \cdot 10^{-3}$
τ_4	3.8	60	7	$2.8 \cdot 10^{-2}$	$8.6 \cdot 10^{-3}$	$8.1 \cdot 10^{-4}$
τ_5	4.1	132	16	$1.2 \cdot 10^{-2}$	$7.4 \cdot 10^{-3}$	$3.9 \cdot 10^{-4}$
$\boldsymbol{\tau_6}$	4.5	289	36	$\mathbf{5.6 \cdot 10^{-3}}$	$6.3 \cdot 10^{-3}$	$1.9 \cdot 10^{-4}$
τ_7	5.1	648	92	0	$5.2 \cdot 10^{-3}$	$9.7 \cdot 10^{-5}$

TABLE 4.1. Influence of time step on performance of algorithm for K_h of (4.2), i.e. averaged number of iterations in projected gradient scheme $\varnothing$, total number of iterations Σ, CPU-run time in seconds, approximation error in final fiber position, fiber elongation Δl. In addition Δl for K_h of (4.3) on the right.

level, the tolerances are set to be $tol_a = 10^{-3}$, $tol_r = 10^{-2}$. Accuracy and efficiency of the whole algorithm are strongly affected by the choice of the time step τ. To get a qualitative impression, we define a sequence of time steps $(\tau_i)_i$, $\tau_i = 2^{-i} \cdot 10^{-3}$, $i = 0, 1, 2, ...$ and determine the fiber behavior for $t^\star = 10^{-3}$, see Figure 4.2 and Table 4.1. In accordance to the theory, we clearly observe the convergence rate of the implicit Euler time discretization as first order; the approximation error is linear in τ which is emphasized by the marked values for τ_0, τ_3, τ_6 in Table 4.1. Considering the computational effort, the averaged number of iterations required in the projected gradient method reduces for smaller τ due to the better initial guess, whereas the total number of projected gradient runs increases. Thus, moderate choices of τ (here: $\tau \approx 10^{-4}$) yield the fewest number of iterations in total and hence the fastest computation. The fiber elongation $\Delta l(t^\star) = \int |\partial_s \mathbf{r}_h^\tau(t^\star)| ds - l \geq 0$ is originated in Lemma 3 and reduces for smaller τ, certainly $\Delta l \to 0$ for $\tau \to 0$. However, its actual magnitude is a consequence of the definition of K_h. Replacing the set of (4.2) by the one of (4.3) where the linear constraint is imposed at the double number of space points ($d = 2(M-1)$) we obtain a better length conservation (smaller Δl); in particular, Δl decreases here linearly in τ, see Figure 4.3. It is worth to mention that the computational effort stays comparably the same when a respective projection $P_{h,2}$ for (4.3) is used. The reason lies in the same underlying finite element space V_h.

We are interested in using the numerical scheme for the simulation of a fiber dynamics. Therefore, it is important that the elongation is restricted over time. Theoretically,

$$\Delta l(t) \leq t\tau l \leq T\tau l, \quad t \in [0, T], \tag{4.6}$$

holds according to Lemma 3. The bound results from the τ-inequality in the definition of K. In our finite element formulation, the inequality constraint is incorporated in K_h by the choice of d

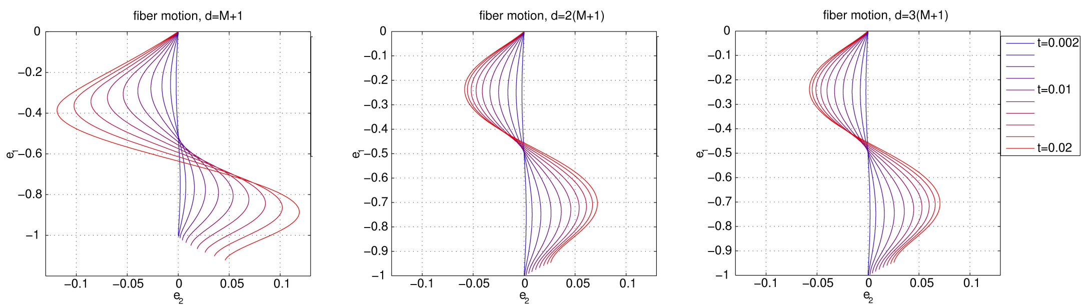

FIGURE 4.4. Fiber dynamics over time for different sets K_h where the linear constraint is satisfied at spatial points σ_j, $j = 1, .., d$. From left to right: $\sigma_j = (j-1)h$ and $d = M-1$ (4.2); $\sigma_j = (j-1)h/2$ and $d = 2(M-1)$ (4.3); $\sigma_j = (j-1)h/3$ and $d = 3(M-1)$.

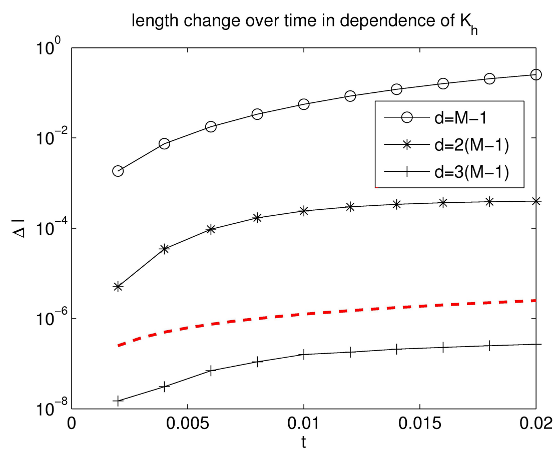

FIGURE 4.5. Elongation over time in dependence of the set K_h, cf. Figure 4.4. The dashed line visualizes the theoretical bound $\Delta l(t) \leq t\tau l$ (4.6).

(Remark 13). This implicit treatment avoids the costly and complex optimization with inequality constraints, but only allows for an a posteriori check of (4.6). We demonstrate the impact of K_h on the fiber behavior for an easy, meaningful scenario: in the stated set-up we apply an additional vertical force, i.e. $\mathbf{f} = f_1\mathbf{e_1} + f_2\mathbf{e_2}$ with f_1 and f_2 of same order. Figure 4.4 shows the fiber motion over time computed with $\tau = \tau_3 \approx 10^{-4}$ for $d \in \{M-1, 2(M-1), 3(M-1)\}$ (here: $f_1 = -10^3\omega$, $f_2(s) = -10^{-2}\sin(2\pi(1-s))$). Whereas K_h with $d = M-1$ (4.2) yields an obviously wrong, disturbing and even growing elongation, the results to K_h with $d = i(M-1)$, $i = 2, 3$ look very similar and reasonable on the first glance. However, only the set K_h where the linear constraint is prescribed at the $3(M-1)$ space points $\sigma_j = (j-1)h/3$ fulfills the theoretical bound (4.6) and provides the desired numerical solution, Figure 4.5. This stays true for the longtime behavior. Note that the computational effort of the algorithm with "Euclidean projections" is determined by time and space discretization (and not by d).

Remark 14. *The commercial software tool FIDYST (Fiber Dynamics Simulation Tool)*[1] *developed by the Fraunhofer Institute of Industrial Mathematics (ITWM) in Kaiserslautern, Germany for the simulation of technical textile production processes computes the fiber dynamics* (2.1) *by help of a full*

[1]FIDYST, Fraunhofer ITWM, Kaiserslautern, `http://www.itwm.fraunhofer.de`

discretization: implicit Euler method in time and finite difference method of higher order in space. The resulting nonlinear system is then solved by a Newton method. In comparison to that approach our proposed scheme is computationally competitive. Moreover, we provide a strict theoretical basis / justification, including convergence result and error bound.

5. Conclusion

In the technical textile industry the dynamics of an elastic inextensible slender fiber is modeled by an nonlinear fourth order partial differential algebraic system. In this paper we have proposed a numerical scheme focusing on an accurate and efficient treatment of the algebraic constraint. Ongoing work deals with the extension of analysis and numerics to the stochastic partial differential algebraic system [15] arising for fibers immersed in turbulent air flows. Here, a stochastic force (source term) of a white noise type is added in the fiber equation. The challenge lies again in the handling of the constraint. So far, the corresponding extensible equation with additive Gaussian noise has been studied in [3].

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